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ON THE SOLUTION OF AN OPTIMAL CONTROL PROBLEM FOR A THIRD ORDER EQUATION

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The problem of the minimum of a quadratic functional with an additional condition at a finite moment is considered for processes described by the initial-boundary value problem for a third-order linear equation with a single spatial variable in the presence of distributed and starting controls.

***Keywords**: distributed control, initial control, third order equation*

Statement of the problem. Let us consider the process described by the problem

$$\beta z_{tt} + z_t - \varepsilon z_{xxt} - z_{xx} = u(x,t), \quad (x,t) \in Q = \{0 < x < 1, 0 < t < t_1\}, \tag{1}$$

$$z(x,0) = 0, \quad z_t(x,0) = \upsilon(x), 0 \le x \le 1, \tag{2}$$

$$z(0,t) = z(1,t) = 0, \quad 0 \le t \le t_1, \tag{3}$$

where β, ε are given sufficiently small positive constants, t_1 is fixed moment of time.

It is required to find such a distributed control $u = u(x,t) \in L_2(Q)$ and start control $\upsilon = \upsilon(x)$, for which the corresponding solution $z(x,t)$ of problem (1)-(3) satisfies condition

$$z(x,t_1) = \varphi(x) \tag{4}$$

and the functional

$$J(u,v) = \iint_Q u^2(x,t)dxdt + \alpha \int_0^l v^2(x)dx \tag{5}$$

takes the smallest value, where α is a positive number.

We note that the solution of problem (1) - (3) is understood in the generalized sense.

Representation of the solution of problem (1) - (3). We seek the solution of this problem in the form $z(x,t) = y(x,t) + \omega(x,t)$, where $y(x,t)$ and $\omega(x,t)$ are the solutions of the following problems, respectively

$$\begin{cases} \beta y_{tt} + y_t - \varepsilon y_{xxt} - y_{xx} = 0, \\ y(x,0) = 0, \quad y_t(x,0) = \upsilon(x), \\ y(0,t) = y(1,t) = 0; \end{cases} \tag{I}$$

$$\begin{cases} \beta \omega_{tt} + \omega_t - \varepsilon \omega_{xxt} - \omega_{xx} = u(x,t), \\ \omega(x,0) = 0, \quad \omega_t(x,0) = 0, \\ \omega(0,t) = \omega(1,t) = 0; \end{cases} \tag{II}$$

We seek the solution of problem (I) by the method of separation of variables, i.e. we seek in the form $y(x,t) = X(x) \cdot T(t)$. Then we obtain an eigenvalue problem with respect to $X(x)$

$$X''(x) + \lambda X(x) = 0, \qquad X(0) = X(1) = 0.$$

It is known that the eigenvalues and orthonormal eigenfunctions of this problem are

$$\lambda_k = \pi^2 k^2, \qquad X_k(x) = \sqrt{2}\sin \pi kx, \ k = 1,2,....$$

Taking this into account with respect to $T(t)$, we obtain the following Cauchy problem

$$\beta \ddot{T} + (1 + \varepsilon\pi^2 k^2)\dot{T} + \pi^2 k^2 = 0, \ T(0) = 0, \ \dot{T}(0) = \upsilon_k. \tag{6}$$

Solving this problem and substituting its solution into $y(x,t) = X(x) \cdot T(t)$, we obtain the following representation

$$y(x,t)=\sum_{k=1}^{\infty}\frac{1}{l_2(k)-l_1(k)}\upsilon_k\cdot\left(e^{l_2(k)t}-e^{l_1(k)t}\right)\sin\pi kx, \quad (7)$$

where $l_1(k), l_2(k)$ are the roots of the characteristic equation corresponding to equation (6),

$$\upsilon_k=\sqrt{2}\int_0^1\upsilon(x)\sin\pi kxdx.$$

We seek the solution of problem (II) in the form

$$\omega(x,t)=\sum_{k=1}^{\infty}R_k(t)\sin\pi kx \ .$$

Then we obtain the Cauchy problem with respect to $R_k(t)$

$$\begin{aligned}&\beta\ddot{R}_k+\left(1+\varepsilon\pi^2k^2\right)\dot{R}_k+\pi^2k^2R_k=\\&=u_k(t),\ R_k(0)=0,\ \dot{R}_k(0)=0,\end{aligned} \quad (8)$$

where

$$u_k(t)=\sqrt{2}\int_0^1 u(\xi,t)\sin\pi k\xi d\xi \ .$$

It is easy to show that

$$R_k(t)=\frac{1}{l_2(k)-l_1(k)}\int_0^t\left(e^{l_2(k)(t-s)}-e^{l_1(k)(t-s)}\right)u_k(s)ds,\ k=1,2,...$$

and therefore the formal solution of problem (II) is

$$\omega(x,t)=\sum_{k=1}^{\infty}\frac{1}{\beta(l_2(k)-l_1(k))}\int_0^t\left(e^{l_2(k)(t-s)}-e^{l_1(k)(t-s)}\right)u_k(s)ds\sin\pi kx. \quad (9)$$

Substituting (7) and (9) into $z(x,t)=y(x,t)+\omega(x,t)$, we get that the formal solution of problem (1)-(3) is

$$.\ z(x,t)=\sum_{k=1}^{\infty}\frac{1}{(l_2(k)-l_1(k))}\left[\upsilon_k\cdot\left(e^{l_2(k)t}-e^{l_1(k)t}\right)+\right.$$

$$\left.+\frac{1}{\beta}\int_0^t\left(e^{l_2(k)(t-s)}-e^{l_1(k)(t-s)}\right)u_k(s)ds\right]\sin\pi kx\ , \quad (10)$$

Further, using the negativity of $l_1(k), l_2(k)$, the orthogonality of systems $\{\sin\pi kx\}$, $\{\cos\pi kx\}$, we prove that the series for $z(x,t)$, representable in the form (10) and the series obtained from (10) by differentiation with respect to t and x converge uniformly in $L_2(0,1)$ with respect to t.

Thus, we have the following

Theorem 1. For any $u(x,t)\in L_2(Q),\ \upsilon(x)\in L_2(0,1)$ problem (1)-(3) has a unique solution representable in the form (10).

Reduction of the minimum problem for the functional (5) to the conditional extremum problem. Substituting (10) in condition (4) after simple transformations, we obtain equalities

$$2\int_0^{t_1}A_k(t)u_k(t)dt+\beta C_k\upsilon_k=2\beta(l_2(k)-l_1(k))\varphi_k,\ k=1,2,..., \quad (11)$$

where

$$A_k(t)=e^{l_2(k)(t_1-t)}-e^{l_1(k)(t_1-t)},\ C_k=e^{l_2(k)t_1}-e^{l_1(k)t_1},$$

$$\varphi_k=\sqrt{2}\int_0^1\varphi(x)\sin\pi kxdx.$$

Substituting the expansions

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)\sin\pi kx\ ,\ \upsilon(x)=\sum_{k=1}^{\infty}\upsilon_k\sin\pi kx$$

into the functional (5) and taking into account the orthonormality of the system $\{\sin\pi kx\}$, we reduce it to the form

$$J(x,\upsilon)=\sum_{k=1}^{\infty}\left[\int_0^{t_1}u_k^2(t)dt+\alpha(\upsilon_k)^2\right]. \quad (12)$$

Thus, the minimization of the functional (5) under condition (4) is reduced to the minimum problem (12) under the conditions (11).

Since the conditions (11) and the terms in (12) do not depend on each other, we can consider the sequence of minima of the functionals

$$J_k(u_k,\upsilon_k)=\int_0^{t_1} u_k^2(t)dt+\upsilon_k^2,\ k=1,2,...,\quad (13)$$

under the conditions (11).

To solve the *k*-th problem, we formulate the Lagrange function

$$L(u_k,\upsilon_k,\lambda_0,\lambda_1)=\lambda_0\left[u_k^2(t)+\alpha\upsilon_k^2\right]+\lambda_1\left[2A_k(t)u_k(t)+\beta C_k\upsilon_k^2\right].$$

In the virtue of [2], we find that if $\tilde{u}_k(t),\tilde{\upsilon}_k^2$ gives the minimum to a functional $J_k(u_k,\upsilon_k)$, then there exist numbers $\tilde{\lambda}_0,\tilde{\lambda}_1$ that are not equal to zero simultaneously that the identities hold true

$$\tilde{\lambda}_0\tilde{u}_k(t)+\tilde{\lambda}_1A_k(t)=0,\quad \tilde{\lambda}_0\alpha\tilde{\upsilon}_k+\tilde{\lambda}_1\beta C_k=0.$$

Since $A_k(t)\neq 0, t\in[0,t_1)$, then assuming $\tilde{\lambda}_0=1$ we determine the following relationships from the last system:

$$\tilde{u}_k(t)=-\tilde{\lambda}_1A_k(t),\quad \tilde{\upsilon}_k=-\tilde{\lambda}_1\frac{\beta}{\alpha}C_k,$$

or after determining $\tilde{\lambda}_1$ from the condition (11) we finally have:

$$\tilde{u}_k(t)=\frac{\alpha\beta(l_2(k)-l_1(k))\varphi_kA_k(t)}{D_k},\quad \tilde{\upsilon}_k=\frac{\beta^2(l_2(k)-l_1(k))\varphi_kC_k}{D_k},\quad (14)$$

where

$$D_k=2\alpha\int_0^{t_1}A_k^2(t)dt+\beta_k^2C_k^2.$$

On the basis of these calculations the following theorem is proved:

Theorem 2. For any $u(x,t)\in L_2(Q),\ \upsilon(x)\in L_2(0,1)$ the problem of the minimum of the functional (5) under the condition (4) has a solution $\tilde{u}_k(x,t),\ \tilde{\upsilon}_k(x)$ that is representable in the form

$$\tilde{u}(x,t)=\sum_{k=1}^{\infty}\tilde{u}_k(t)\sin\pi kx,\quad \tilde{\upsilon}(x)=\sum_{k=1}^{\infty}\tilde{\upsilon}_k\sin\pi kx,$$

where $\tilde{u}_k(t),\ \tilde{\upsilon}_k$ are represented by formulas (14).

The theorem is proved on the basis of the following arguments: if $\tilde{u}_k(t),\ \tilde{\upsilon}_k$ gives the minimum to the functional (5) under the condition (4), then the function

$$J_k(\delta,\gamma)=\int_0^{t_1}\left[\tilde{u}_k^2(t)+\delta h_k(t)\right]^2dt+\alpha\left(\tilde{\upsilon}_k+\gamma r_k\right)^2\quad (15)$$

attains a minimum at the point (0,0) under the condition

$$g(\delta,\gamma)=2\int_0^{t_1}A_k(t)\left[\tilde{u}_k(t)+\delta h_k(t)\right]dt+$$

$$+\beta C_k\left(\tilde{\upsilon}_k+\gamma r_k\right)-2\beta\left(l_2(k)-l_2(k)\right)=0.\quad (16)$$

We make the Lagrangian for problem (15), (16)

$$L(\delta,\gamma,\lambda_0,\lambda_1)=\lambda_0\left[\int_0^{t_1}\left(\tilde{u}_k^2(t)+\delta h_k(t)\right)^2dt+\alpha\left(\tilde{\upsilon}_k+\gamma r_k\right)^2\right]+$$

$$+\lambda_1\left[2\int_0^{t_1}A_k\left(\tilde{u}_k(t)+\delta h_k(t)\right)dt+\beta C_k\left(\tilde{\upsilon}_k+\gamma r_k\right)-2\beta\left(l_2(k)-l_1(k)\right)\right]$$

and show that the matrix composed of the second derivatives of the function $L(\delta,\gamma,\lambda_0,\lambda_1)$ with respect to δ,γ at (0,0) under the condition

$$\frac{\partial g(0,0)}{\partial\delta}a_1+\frac{\partial g(0,0)}{\partial\gamma}a_2=0$$

is positive, which means that the point (0,0) is a minimum point of $J_k(\delta,\gamma)$. Hence, $\tilde{u}_k(t),\tilde{\upsilon}_k$ delivers a minimum to the functional $J_k(u_k,\upsilon_k)$ under the condition (11).

On the convergence of series in (12). To prove the convergence of these series at $\tilde{u}_k(t),\ \tilde{\upsilon}_k$, we show that

$$\tilde{u}_k^2(t)\sim Me^{2l_1(k)},\quad \tilde{\upsilon}_k\sim Me^{2l_1(k)}$$

and since $l_1(k)<0$, then series $\sum_{k=1}^{\infty}\tilde{u}_k^2(t)$ is convergent uniformly with respect to t, and the number series $\sum_{k=1}^{\infty} \tilde{\upsilon}_k^2$ is convergent.

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ÜÇTƏRTİBLİ TƏNLİK ÜÇÜN BİR OPTİMAL İDARƏETMƏ MƏSƏLƏSİNİN HƏLLİ HAQQINDA

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Məqalədə bir fəza dəyişəni olan üçtərtibli xüsusi törəməli xətti tənlik üçün qarışıq məsələ ilə təsvir olunan prosesin son anında əlavə şərt daxilində kvadratik funksionala minimum verən paylanmış və başlanğıc idarəedicilərin tapılması məsələsi araşdırılır.

Açar sözlər: *paylanmış idarəedici, başlanğıc idarəedici, üçtərtibli tənlik*

О РЕШЕНИИ ОДНОЙ ЗАДАЧИ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ ДЛЯ УРАВНЕНИЯ ТРЕТЬЕГО ПОРЯДКА

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В работе рассматривается задача минимума квадратичного функционала с дополнительным условием в конечный момент времени в процессах, описываемых начально-краевой задачей для линейного уравнения третьего порядка с одной пространственной переменной при наличии распределенного и стартового управлений.

Ключевые слова: *распределенное управление, стартовое управление, уравнение третьего порядка*