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ASYMPTOTICS OF THE SOLUTION OF A BOUNDARY VALUE PROBLEM FOR A BISINGULARLY PERTURBED ONE-CHARACTERISTIC DIFFERENTIAL EQUATION

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The first terms of the asymptotics of the solution of a boundary value with regard to inner layer arising near the bisectrix of the first quadrant were constructed.

Keywords: *asymptotics, boundary layer type function, remainder term*

Introduction

Much less works were devoted to the study of singularly perturbed non-classical differential equations compared with the works belonging to classic equations. Such equations very often arise when studying different phenomena with non-uniform transitions from one physical characteristics to another ones. In [1], M.I. Vishik and L.A. Lusternik have introduced the so-called one-characteristic differential equations. The differential equations of $2k+1$ even order $L_{2k+1}u \equiv A_1(A_{2k}u) + B_{2k}u = f$ are called one-characteristical if the operator A_1 is of first order, A_{2k} is an elliptic operator of order $2k$, and B_{2k} is any differential operator of order at most $2k$. As is known, the boundary layer phenomena arise not only near the boundary of the considered domain, but also interior to the domain. This occurs when along some manifold the solution of the degenerate problem has some discontinuity or discontinuity of its derivatives, absent in the solutions of the input problem. In the case when the corresponding degenerate problem has non-smooth solution, by A.M. Il'in terminology these problems are called bisingular problems. In this paper, the first terms of the asymptotics of a boundary value problem are constructed for a third order bisingularly perturbed one-characteristic differential equation degenerated into one-characteristic equation of first order with regard to a boundary layer function near some line interior to the considered domain. Earlier, for constructing the asymptotics of bisingularly perturbed problems the combination method was used, and the method of boundary functions was not used directly.

Statement of the problem and solution of the degenerate problem

In $D = \{(x,y) | 0 \le x \le 1,\ 0 \le y \le 1\}$ we consider the following boundary value problem

$$L_\varepsilon u \equiv \varepsilon^2 \frac{\partial}{\partial x}(\Delta u) - \varepsilon \Delta u + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + F(x,y,u) = 0, \quad (1)$$

$$u\big|_{x=0} = 0,\ u\big|_{x=1} = 0,\ \frac{\partial u}{\partial x}\big|_{x=1} = 0, \quad (0 \le y \le 1), \quad (2)$$

$$u\big|_{y=0} = 0, \quad u\big|_{y=1} = 0, \quad (0 \le x \le 1), \quad (3)$$

where $\varepsilon > 0$ is a small parameter, Δ is a Laplace operator, $F(x,y,u)$ is a given smooth function.

For $\varepsilon = 0$ equation (1) degenerates into the equation

$$\frac{\partial W}{\partial x} + \frac{\partial W}{\partial y} + F(x,y,W) = 0. \quad (4)$$

For degenerate equation (4) we give the boundary conditions

$$W\big|_{x=0} = 0,\ (0 \le y \le 1);\ W\big|_{y=0} = 0,\ (0 \le x \le 1). \quad (5)$$

It is easy to see that the solution of boundary value problem (4), (5) is continuous everywhere in D, but the first derivatives of the function $W(x,y)$ may have first kind discontinuities on characteristic $y=x$ of equation (6) passing from the origin of coordinates. It is necessary to construct a boundary layer function near the line $y=x$ to compensate this discontinuity.

Constructing boundary layer functions

We make a change of variables $x_1 = x+y,\ y_1 = y-x$ and write the operator L_ε in new coordinates

$$L_{\varepsilon,1}\eta \equiv 2\varepsilon^2\left(\frac{\partial^3\eta}{\partial x_1^3} - \frac{\partial^3\eta}{\partial x_1^2\partial y_1} + \frac{\partial^3\eta}{\partial x_1\partial y_1^2} - \frac{\partial^3\eta}{\partial y_1^3}\right) - $$
$$-2\varepsilon\left(\frac{\partial^2\eta}{\partial x_1^2} + \frac{\partial^2\eta}{\partial y_1^2}\right) + 2\frac{\partial\eta}{\partial x_1} + F\left(\frac{x_1}{2} - \frac{y_1}{2},\ \frac{x_1}{2} + \frac{y_1}{2},\ \eta\right).$$

Introducing local coordinates $x_1 = x_1,\ y_1 = \varepsilon\xi$ and looking for a boundary layer function near the line $y=x$ in the form $\eta = \varepsilon(\eta_0 + \varepsilon\eta_1 + ...)$ as an approximate solution of the equation $L_{\varepsilon,1}(W+\eta) - L_{\varepsilon,1}W = 0(\varepsilon)$, in the first approximation we get the following equation to determine η_0:

$$\frac{\partial^3\eta_0}{\partial\xi^3} + \frac{\partial^2\eta_0}{\partial\xi^2} = 0. \qquad (6)$$

Obviously, the function $\eta_0 = \varphi(x_1)\left(e^{-\xi} - 1\right)$ satisfies equation (6). We will consider it the solution of equation (6) only for $\xi > 0$. For $\xi < 0$ as η_0 we take the solution $\eta_0 \equiv 0$. The function $\varphi(x_1)$ is determined from the condition that the function $\frac{\partial}{\partial x}(W+\eta)$ is continuous on the line $y=x$ and has the form

$$\varphi(x_1) = -\left(\frac{\partial W}{\partial x}\Big|_{y-x=+0} - \frac{\partial W}{\partial x}\Big|_{y-x=-0}\right).$$

It is easily verified that the function $\frac{\partial}{\partial y}(W+\eta)$ will also be a continuous function on the line $y=x$. We multiply the function η by the smoothing function and preserve its previous notation.

The function $W+\eta$, generally speaking, does not satisfy the boundary conditions on the boundaries $\Gamma_1 = \{(x,y)|x=1\ 0\le y\le 1\}$ and $\Gamma_2 = \{(x,y)|\ 0\le x\le 1,\ y=1\}$. Therefore, it is necessary to construct the boundary layer functions V and ψ near the boundaries Γ_1 and Γ_2. We will not dwell on construction of boundary layer functions near the boundaries Γ_1 and Γ_2.

The boundary layer function near the boundary Γ_1 is sought in the form $V = V_0 + \varepsilon V_1$ as an approximate solution of the equation $L_{\varepsilon,2}(W+\eta+V) - L_{\varepsilon,2}(W+\eta) = 0(\varepsilon)$, where $L_{\varepsilon,2}$ denotes a new decomposition of the operator L_ε near Γ_1. To determine the functions V_0 and V_1 we get the equation:

$$\frac{\partial^3 V_0}{\partial t^3} + \frac{\partial^2 V_0}{\partial t^2} + \frac{\partial V_0}{\partial t} = 0, \qquad (7)$$

$$\frac{\partial^3 V_1}{\partial t^3} + \frac{\partial^2 V_1}{\partial t^2} + \frac{\partial V_1}{\partial t} = \frac{\partial V_0}{\partial y} + \frac{\partial F(1,y,W+\eta)}{\partial u}V_0, \qquad (8)$$

where $t = (1-x)/\varepsilon$. The boundary conditions for equations (7), (8) are found from the requirement

$$(W+\eta+V)\Big|_{x=1} = 0,\ \frac{\partial}{\partial x}(W+\eta+V)\Big|_{x=1} = 0. \qquad (9)$$

When the function $F(x,y,u)$ satisfies the condition $F(1,0,u)=0$, then in addition to conditions (9) the constructed sum satisfies also the condition

$$(W+\eta+V)\Big|_{y=0} = 0. \qquad (10)$$

We multiply the function V by the smoothing function. Then the sum $W+\eta+V$ will satisfy also the condition

$$(W+\eta+V)\big|_{x=0}=0. \qquad (11)$$

Then we construct a boundary layer type function ψ near the boundary Γ_2. The function ψ is sought in the form $\psi=\psi_0+\varepsilon\psi_1$, as the solution of the equation $L_{\varepsilon,3}(W+\eta+V+\psi)-L_{\varepsilon,3}(W+\eta+V)=0(\varepsilon)$, where $L_{\varepsilon,3}$ denotes the decomposition of the operator L_ε near Γ_2. The functions ψ_0 and ψ_1 are determined from the equation

$$\frac{\partial^2\psi_0}{\partial\tau^2}+\frac{\partial\psi_0}{\partial\tau}=0, \qquad (12)$$

$$\frac{\partial^2\psi_1}{\partial\tau^2}+\frac{\partial\psi_1}{\partial\tau}=\frac{\partial^3\psi_0}{\partial x\partial\tau^2}+\frac{\partial\psi_0}{\partial x}+\frac{\partial F(x,1,W+\eta+V)}{\partial u}\psi_0, \qquad (13)$$

where $\tau=\dfrac{1-y}{\varepsilon}$. The boundary condition for equations (12), (13) are found from the requirement

$$(W+\eta+V+\psi)\big|_{y=1}=0. \qquad (14)$$

We multiply the function ψ by a smoothing function. In addition to condition (14), the constructed sum satisfies also the condition

$$(W+\eta+V+\psi)\big|_{y=0}=0. \qquad (15)$$

If $F(0,1,u)=0$, then in addition to condition (14), (15), the constructed function will satisfy the following conditions

$$(W+\eta+V+\psi)\big|_{x=0}=0,\quad (W+\eta+V+\psi)\big|_{x=1}=$$
$$=0,\ \frac{\partial}{\partial x}(W+\eta+V+\psi)\big|_{x=1}=0. \qquad (16)$$

Estimating the remainder term and the main result

We introduce the denotation

$$z=u-(W+\eta+V+\psi) \qquad (17)$$

and call the function z a remainder term. It is easy to show that the function z for $x\neq y$ is a solution of the following boundary value problem

$$L_\varepsilon z=\varepsilon h(\varepsilon,x,y), \qquad (18)$$

$$z\big|_{x=0}=z\big|_{x=1}=0,\ \frac{\partial z}{\partial x}\Big|_{x=1}=0,\ z\big|_{y=0}=z\big|_{y=1}=0, \qquad (19)$$

where $h(\varepsilon,x,y)$ is a known function.

It is easy to show that for the solution of boundary value problem (18), (19), the following estimations are valid

$$\|z\|_{L_2(D)}\le C\varepsilon,\quad \left\|\frac{\partial z}{\partial x}\right\|_{L_2(D)}+\left\|\frac{\partial z}{\partial y}\right\|_{L_2(D)}\le C, \qquad (20)$$

where $c=const>0$ is independent of ε.

Finding u from (17), we have

$$u=W+\eta+V+\psi+z. \qquad (21)$$

Joining the obtained results we arrive at the following statement.

Theorem. If $F(x,y,u)\in C^3(D\times(-\infty,+\infty))$, $F(1,0,u)=0,\ F(0,1,u)=0$, then for the solution of problem (1)-(3) in the first approximation it holds asymptotic representation (21), where W is the solution of the degenerate problem, $\eta=\varepsilon\eta_0$ is a boundary layer near the line $y=x$, $V=V_0+\varepsilon V_1$ and $\psi=\psi_0+\varepsilon\psi_1$ are boundary layer functions near the boundaries Γ_1 and Γ_2, z is a remainder term, and estimation (20) is valid for it.

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BİSİNQULYAR HƏYƏCANLANMIŞ BİRXARAKTERİSTİKALI DİFERENSİAL TƏNLİK ÜÇÜN QOYULMUŞ SƏRHƏD MƏSƏLƏSİNİN HƏLLİNİN ASİMPTOTİKASI

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Bisinqulyar həyəcanlanmış birxarakteristikalı diferensial tənlik üçün qoyulmuş sərhəd məsələsinin həllinin birinci rübün tənböləni yaxınlığında daxili sərhəd zolaq tipli funksiya qurulmaqla asimptotik ayrılışının ilk hədləri qurulmuşdur.

Açar sözlər: *asimptotika, sərhəd zolaq tipli funksiya, qalıq həddi*

АСИМПТОТИКА РЕШЕНИЯ КРАЕВОЙ ЗАДАЧИ ДЛЯ БИСИНГУЛЯРНО ВОЗМУЩЕННОГО ОДНОХАРАКТЕРИСТИЧЕСКОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ

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Построены первые члены асимптотики решения краевой задачи для бисингулярно возмущенного одно-характеристического дифференциального уравнения с учетом внутреннего слоя, возникающего вблизи биссектрисы первого квадранта.

Ключевые слова: *асимптотика, функция типа пограничного слоя, остаточный член*