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BASIS PROPERTIES OF THE SYSTEM OF EXPONENTIALS IN WEIGHTED MORREY SPACES

Corresponding Member of ANAS B.T.Bilalov, A.A.Huseynli, Z.A.Kasumov

In this paper the basis properties (completeness, minimality and basicity) of the system of exponentials are investigated in weighted Morrey spaces, where the weight function is defined as a product of power functions. Although the same properties of the system of exponentials, as well as their pertubations, are well studied in weighted Lebesgue spaces, the situation changes cardinally in Morrey spaces, since Morrey spaces are not separable. Nevetheless, there are works, that study these problems. In [1-3] there were studied the basis properties of the system of exponentials in Morrey space. Some approximation problems have been investigated in Morrey-Smirnov classes in [4].

Keywords: *Morrey space, basis properties, system of exponentials*

We study the basis properties of the system $\{e^{int}\}_{n\in Z}$ in weighted Morrey space $L_\nu^{p,\lambda}(-\pi,\pi)$, where the weight function $\nu(t)$ is taken as

$$\nu(t)=\prod_{k=0}^{r}\left|t-t_k\right|^{\alpha_k},\quad t\in[-\pi,\pi], \qquad (1)$$

t_k are any point in the interval $[-\pi,\pi]$, and $\alpha_k\in R$ for all $k=0,1,...,r$.

Recall that, for $1<p<\infty$ and $0\le\lambda<1$ the Morrey space $L^{p,\lambda}(a,b)$ is defined as the set of measurable functions f on (a,b) such that

$$\|f\|_{p,\lambda}:=\sup_{I\subset(a,b)}\left(\frac{1}{|I|^{\lambda}}\int_I\left|f(t)\right|^p dt\right)^{\frac{1}{p}}<\infty,$$

where $I\subset(a,b)$ is any interval. It is clear that $L^{p,\lambda}(a,b)$ are Banach spaces. For Morrey spaces in multidimensional euclidean spaces, their generalizations and their comprehensive study see, for examle, [5;6]. The $L_p(a,b)$ spaces with the Lebesgue measure correspond with the case $\lambda=0$. The weighted Morrey space $L_\nu^{p,\lambda}(a,b)$ is defined in the usual way

$$L_\nu^{p,\lambda}(a,b):=\left\{f:\nu f\in L^{p,\lambda}(a,b)\right\},$$

with $\|f\|_{p,\lambda;\nu}:=\|\nu f\|_{p,\lambda}$. It is evident that the space $L_\nu^{p,\lambda}(a,b)$ contains constant functions if and only if $\nu\in L^{p,\lambda}(a,b)$. Throughout the paper, unless otherwise stated, we will assume that $1<p,q<\infty$, $p^{-1}+q^{-1}=1$ and $0<\lambda<1$.

The following lemma has been proved by Samko [7] in the case of Morrey space on a bounded rectifiable curve. In our case it reads

Lemma 1. *The power function $|t-t_0|^{\alpha}$, $t_0\in[a,b]$, belongs to the Morrey space $L^{p,\lambda}(a,b)$ if and only if $\alpha\in\left[\frac{\lambda-1}{p},\infty\right)$.*

Direct application of the above lemma implies the following

Proposition 1. *Let the weight function ν be given as in (1). Then $\{e^{int}\}_{n\in Z}\subseteq L_\nu^{p,\lambda}(-\pi,\pi)$ if and only if*

$$\alpha_k\in\left[\frac{\lambda-1}{p},\infty\right),\ \text{for all}\ k=0,1,2,...,r. \qquad (2)$$

Let us introduce the following linear space of measurable functions

$$\left(L^{p,\lambda}\right)'=\left\{g:\sup_{\|f\|_{p,\lambda}=1}\|f\,g\|_{L_1}<+\infty\right\}.$$

Lemma 2.

$$|t|^{\beta} \in \left(L^{p,\lambda}(-\pi,\pi)\right)' \Leftrightarrow \beta \in \left(-\frac{\lambda-1}{p}-1,\infty\right),\ 0 \le \lambda < 1, 1 < p < +\infty.$$

Indeed, first suppose that $\beta \in \left(-\frac{\lambda-1}{p}-1,\infty\right)$. Then, for all $f \in L^{p,\lambda}(-\pi,\pi)$ we have

$$\int_{-\pi}^{\pi} |t|^{\beta} |f(t)| dt = \sum_{k=1}^{\infty} \int_{|t| \in \left[2^{-k}\pi, 2^{-k+1}\pi\right]} |t|^{\beta} |f(t)| dt \le$$

$$\le c \sum_{k=1}^{\infty} 2^{-k\beta} \int_{|t| \in \left[2^{-k-1}\pi, 2^{-k}\pi\right]} |f(t)| dt \le$$

$$\le c \sum_{k=1}^{\infty} 2^{-k\beta} 2^{-k\left(1-\frac{1}{p}\right)} \left(\int_{|t| \in \left[-2^{-k}\pi, 2^{-k}\pi\right]} |f(t)|^{p} dt \right)^{\frac{1}{p}} =$$

$$= c \sum_{k=1}^{\infty} 2^{-k\left(\beta+1-\frac{1}{p}+\frac{\lambda}{p}\right)} \|f\|_{L^{p,\lambda}} \le$$

$$\le c \|f\|_{L^{p,\lambda}}.$$

Then, $|t|^{\beta} \in \left(L^{p,\lambda}(-\pi,\pi)\right)'$.

Conversely, suppose that $\beta \notin \left(-\frac{\lambda-1}{p}-1,\infty\right)$. That is $\beta + \frac{\lambda-1}{p} \le -1$. Then, $|t|^{\frac{\lambda-1}{p}} \in L^{p,\lambda}(-\pi,\pi)$ and

$$\int_{-\pi}^{\pi} |t|^{\beta} |t|^{\frac{\lambda-1}{p}} dt = \int_{-\pi}^{\pi} |t|^{\beta+\frac{\lambda-1}{p}} dt = \infty.$$

Thus $|t|^{\beta} \notin \left(L^{p,\lambda}\right)'$. This completes the proof.

Let $f(\cdot)$ be the given function on $[a,b]$. In determining the Zorko type subspace we will assume that the function $f(\cdot)$ is continued to $[2a-b, 2b-a]$ with the following expression (and this function is also denoted by $f(\cdot)$)

$$f(x) = \begin{cases} f(2a-x), x \in [2a-b, a), \\ f(2b-x), x \in (b, 2b-a]. \end{cases}$$

So, following Zorko [8], we consider the subspace

$$\tilde{L}_{\nu}^{p,\lambda}(a,b) := \left\{ f \in L_{\nu}^{p,\lambda}(a,b) : \| f(\cdot+\delta) - f(\cdot) \|_{p,\lambda;\nu} \to 0 \text{ as } \delta \to 0 \right\},$$

where ν is given as in (1) under conditions (2). We will refer to this subspace as the Zorko subspace of $L_{\nu}^{p,\lambda}(a,b)$. Also, we consider the $L_{\nu}^{p,\lambda}$ -closure of $\tilde{L}_{\nu}^{p,\lambda}(a,b)$ and denote it by $M_{\nu}^{p,\lambda}(a,b)$. It is easy to see that if $\nu \in L^{p,\lambda}(a,b)$, then $C[-a,b] \subset M_{\nu}^{p,\lambda}(a,b)$. In fact, let $f \in C[a,b]$ be an arbitrary function and δ be an arbitrary number (with sufficiantly small $|\delta|$). It is obvious that the extended function $f(\cdot)$ is continuous on $[2a-b, 2b-a]$. We have

$$\| f(\cdot+\delta) - f(\cdot) \|_{p,\lambda,\nu} = \sup_{I \subset (a,b)} \left(\frac{1}{|I|^{\lambda}} \int_I \left| \left(f(t+\delta) - f(t) \right) \nu(t) \right|^p dt \right)^{1/p} \le$$

$$\le \sup_{t \in [a,b]} \left| f(t+\delta) - f(t) \right| \|\nu\|_{p,\lambda} \to 0,\ \delta \to 0.$$

Thus, we have the following

Lemma 3. *Let* $\nu \in L^{p,\lambda}(a,b)$. *Then,* $C[a,b] \subset M_{\nu}^{p,\lambda}(a,b)$.

The following theorem shows that the subspace $M_{\nu}^{p,\lambda}(a,b)$ is in fact, “natural” to be studied the basis properties in.

Theorem 1. *Let* ν *be given as in (1) and*

$$\alpha_k \in \left[-\frac{1-\lambda}{p}, -\frac{1-\lambda}{p}+1 \right),\ k = \overline{0,r}.$$

Then the set $C^{\infty}[-\pi,\pi]$ *is dense in* $M_{\nu}^{p,\lambda}(-\pi,\pi)$.

Special relevant for our purpose is the boundedness of the singular integral operators in Morrey spaces. In [7], the following result was proved.

Theorem 2. *Let* $0 \le \lambda < 1,\ 1 < p < +\infty$, *and* ν *be given as in (1). The singular integral operator* S

$$Sf(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(t)}{e^{it}-e^{ix}}dt, \qquad x \in (-\pi, \pi)$$

is bounded in the space $L_{\nu}^{p,\lambda}(-\pi,\pi)$ *if and only if*

$$\alpha_k \in \left(\frac{\lambda-1}{p}, \frac{1-\lambda}{q}+\lambda\right), \text{for all } k = 0,1,2,..., r.$$

Using this theorem we prove our main

Theorem 3. *Let* ν *be given as in (1).*

(I) *The system* $\{e^{int}\}_{n\in Z}$ *is minimal in* $L_{\nu}^{p,\lambda}(-\pi,\pi)$ *if*

$$\alpha_k \in \left[\frac{\lambda-1}{p}, \frac{1-\lambda}{q}+\lambda\right) \text{ for all } k = 0,1,..., r \;;$$

(II) *The system* $\{e^{int}\}_{n\in Z}$ *is complete in* $M_{\nu}^{p,\lambda}(-\pi,\pi)$ *if*

$$\alpha_0;\alpha_r \in \left(-\frac{1-\lambda}{p}, -\frac{1-\lambda}{p}+1\right), \alpha_k \in \left[-\frac{1-\lambda}{p}, -\frac{1-\lambda}{p}+1\right), \; k=\overline{1,r-1} \;;$$

(III) *The system* $\{e^{int}\}_{n\in Z}$ *forms a basis in* $M_{\nu}^{p,\lambda}(-\pi,\pi)$ *if and only if*

$$\alpha_k \in \left(\frac{\lambda-1}{p}, \frac{1-\lambda}{q}+\lambda\right), \; \textit{for all } k = 0,1,2,..., r.$$

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İnstitute of Mathematics and Mechanics of ANAS
b_bilalov@mail.ru
alihuseynli@gmail.com

EKSPONENSİAL SİSTEMLƏRİN ÇƏKİLİ MORRİ FƏZALARINDA BAZİSLİK XASSƏLƏRİ

B.T.Bilalov, Ə.A.Hüseynli, Z.Ə.Qasımov

Məqalədə eksponensial sistemlərin çəki funksiyası qüvvət funksiyası olduğu halda, çəkili Morri fəzalarında bazislik (tamlıq, minimallıq, bazislik) xassələri öyrənilir. Eksponensial sistemlərin (o cümlədən onların həyəcanlanmalarının) analoji xassələrinin çəkili Lebeq fəzalarında yaxşı öyrənilməsinə baxmayaraq, Morri fəzalarında vəziyyət köklü olaraq dəyişir, çünki bu fəzalar separabel deyil. Buna baxmayaraq eksponensial sistemlərin və onların bəzi həyəcanlanmalarının adi Morri fəzalarında bazisliyi daha əvvəlki [1-3] işlərində öyrənilib. Approksimasiyanın bəzi məsələləri Morri-Smirnov siniflərində [4] işində araşdırılmışdır.

Açar sözlər: *Morri fəzası, bazislik xassələri, eksponensial sistem*

БАЗИСНЫЕ СВОЙСТВА СИСТЕМ ЭКСПОНЕНТ В ВЕСОВЫХ ПРОСТРАНСТВАХ МОРРИ

Б.Т.Билалов, А.А.Гусейнли, З.А.Касумов

В данной статье изучаются базисные свойства (полнота, минимальность, базисность) системы экспонент в весовых пространствах Морри, когда вес имеет степенной вид. Несмотря на то, что аналогичные свойства системы экспонент (в том числе ее возмущения) хорошо изучены в весовых лебеговых пространствах, ситуация в пространствах Морри кардинально изменяется, так как эти пространства не являются сепарабельными. Есть работы, которые изучают эти проблемы. Следует отметить, что базисность системы экспонент и ее некоторые возмущения в обычных пространствах Морри ранее были изучены в работах [1-3]. Некоторые проблемы аппроксимации были исследованы в классах Морри-Смирнова в [4].

Ключевые слова: *пространство Морри, базисные свойства, система экспонент*