UDC 517.984

INVERSE SPECTRAL PROBLEM FOR PERTURBED HARMONIC OSCILLATOR

A.Kh.Khanmamedov[1,2], H.M.Masmaliyev[2], M.H.Mahmudova[2]
(Presented by Academician of ANAS M.F.Mehdiyev)

An inverse spectral problem for anharmonic equation on the half line is considered. Using the transformation operator special solution with asymptotic behavior at infinity is constructed. The main equation of inverse problem is obtained. We develop an algorithm for solving the inverse problem.

Keywords: *the perturbed harmonic oscillator, transformation operator, inverse problem, main equation*

One of the most important problems of wave mechanics is the problem on quant oscillator. Description of oscillating motions of atoms molecules and crystals (see $[1]$). Let as consider a boundary value problem generated by the anharmonic equation

$$-y''+x^2y+q(x)y=\lambda y,\ 0<x<\infty,\quad \lambda\in C, \quad (1)$$

with the boundary condition

$$y(0)=0, \quad (2)$$

where a real-valued potential $q(x)$ satisfies the condition

$$q(x)\in C^{(1)}(-\infty,\infty),\ \int_{-\infty}^{\infty}\left|x^j q(x)\right|dx<\infty,\ j=0,1,2. \quad (3)$$

If condition (3) is satisfied, then problem (1) - (2) has a purely discrete spectrum consisting (see, for example, $[2]-[6]$) of simple eigenvalues $\lambda_n, n=0,1,...$, where $\lambda_n\to+\infty$ as $n\to\infty$. In this case, the corresponding eigenfunctions $\left\{\frac{f(x,\lambda_n)}{\alpha_n}\right\}_{n=0}^{\infty}$, where $\alpha_n=\sqrt{\int_0^{\infty}\left|f(x,\lambda_n)\right|^2dx}$ form an orthonormal basis in space $L_2(0,\infty)$.

In the present work, the inverse spectral problem for the problem (1) - (2) is investigated by the transformation operators method, i.e. the problem of reconstructing a perturbation $q(x)$ from spectral data $\{\lambda_n,\alpha_n>0\}_{n=0}^{\infty}$. As is known to us, the latter problem has not been considered before.

We note that inverse spectral problems for an anharmonic oscillator, i.e. for equation (1), were studied in various contexts by many authors (see $[2]-[8]$, also the bibliography in them).

We consider the unperturbed equation

$$-y''+x^2y=\lambda y,\ 0<x<\infty,\quad \lambda\in C. \quad (4)$$

It is known that equation (4) has $[2],[7]$ the solution $f_0(x,\lambda)=D_{\frac{\lambda}{2}-\frac{1}{2}}(\sqrt{2}x)$, where $D_\nu(x)$ is Weber function. Moreover, (see $[9],[10]$) for each x the function $f_0(x,\lambda)$ is entire and the following asymptotic is fulfilled:

$$f_0(x,\lambda)=\left(\sqrt{2}x\right)^{\frac{\lambda-1}{2}}e^{-\frac{x^2}{2}}\left(1+O\left(x^{-2}\right)\right),\quad x\to\infty, \quad (5)$$

uniformly with respect to λ on bounded domains. It is shown in the works $[2],[7]$ that the spectrum of problem (4), (2) consists of simple eigenvalues $\lambda_n^0=4n+3, n=0,1,...$. In this case, the functions $\{f_0(x,\lambda_n^0)\}_{n=0}^{\infty}$ form an orthogonal basis in space $L_2(0,\infty)$. We have the equalities

$$f_0(x,\lambda_n^0)=D_{2n+1}(\sqrt{2}x)=2^{-\left(n+\frac{1}{2}\right)}e^{-\frac{x^2}{2}}H_{2n+1}(x),$$

where $H_n(x)$ is the Hermite polynomial. From the well-known properties of Hermite polynomials it follows that

$$\left(\alpha_n^0\right)^2 = \int_0^\infty \left| f_0\left(x,\lambda_n^0\right)\right|^2 dx = (2n+1)!\frac{\sqrt{\pi}}{2}.$$

Consequently, the system of functions $\left\{\frac{f_0\left(x,\lambda_n^0\right)}{\alpha_n^0}\right\}_{n=0}^\infty$ serves as an orthonormal basis in the space $L_2(0,\infty)$.

Secondly, the perturbed equation (1) under condition (3) has a solution $f(x,\lambda)$ with asymptotic $f(x,\lambda)= f_0(x,\lambda)(1+o(1)), x\to\infty$. Let as suppose that

$$\sigma(x)=\int_x^\infty |q(t)|dt, \sigma_1(x)=\int_x^\infty \sigma(t)dt .$$

In the following theorem the representation of solution $f(x,\lambda)$ is found by means of transformation operator.

Theorem 1. *If potential $q(x)$ satisfies condition (3) then equation (1) for all λ has unique solution $f(x,\lambda)$ in the form*

$$f(x,\lambda)= f_0(x,\lambda)+\int_x^\infty K(x,t)\ f_0(t,\lambda)dt, \quad (6)$$

where $K(x,t)$ is real continuous function, and

$$|K(x,t)|\le \frac{1}{2}\sigma\left(\frac{x+t}{2}\right)e^{\sigma_1\left(\frac{x+t}{2}\right)}, \quad (7)$$

$$K(x,x)=\frac{1}{2}\int_x^\infty q(t)dt. \quad (8)$$

By (5), (6), the function $f(x,\lambda)$ for each λ belongs to the space $L_2(0,\infty)$. It follows that the spectrum of problem (1) - (2) coincides with the set of zeros of the function $f(0,\lambda)$, i.e. equalities $f(0,\lambda_n)=0, n=0,1,...$ hold true. From the results of the work [2],[7] it follows that the asymptotic equalities hold true:

$$\lambda_n=\lambda_n^0+O\left(\frac{1}{\sqrt{n}}\right), \alpha_n^{-2}=\left(\alpha_n^0\right)^{-2}\left[1+O\left(\frac{\ln n}{\sqrt{n}}\right)\right]. \quad (9)$$

Moreover, on holding true the condition $0\le x^2\le\left(\frac{\lambda}{4}\right)^{\frac{1}{2}-\varepsilon}, \lambda\ge\lambda_0>0, 0<\varepsilon<\frac{1}{3}$ the asymptotic expansion holds true

$$f(x,\lambda)=2^{\frac{\lambda-1}{4}}\pi^{-\frac{1}{2}}\Gamma\left(\frac{\lambda+1}{4}\right)\times$$
$$\times\left\{\cos\left[\pi\frac{\lambda-1}{4}-\sqrt{\lambda}x\right]+\lambda^{-\frac{1}{2}}\left(2x^3+2^{-\frac{1}{2}}\right)O(1)\right\}. \quad (10)$$

Suppose

$$F(x,y)=\sum_{n=0}^\infty\left\{\frac{f_0(x,\lambda_n)f_0(y,\lambda_n)}{(\alpha_n)^2}-\frac{f_0\left(x,\lambda_n^0\right)f_0\left(y,\lambda_n^0\right)}{\left(\alpha_n^0\right)^2}\right\}. \quad (11)$$

By means of (9), (10) it is established that for any fixed x function $F(x,\cdot)$ belongs to space $L_2(0,\infty)$. Further, taking into account that system of functions $\left\{\frac{f_0\left(x,\lambda_n^0\right)}{\alpha_n^0}\right\}_{n=0}^\infty$ and $\left\{\frac{f(x,\lambda_n)}{\alpha_n}\right\}_{n=0}^\infty$ serve as orthonormal bases in space $L_2(0,\infty)$ the following theorem is proved.

Theorem 2. *For each fixed x the function $K(x,y)$, defined in (6), satisfy the integral equation*

$$F(x,y)+K(x,y)+\int_x^\infty K(x,t)F(t,y)dt=0,\ y>x. \quad (12)$$

Using the fact that the system of functions $\left\{\frac{f_0(x,\lambda_n)}{\alpha_n}\right\}_{n=0}^\infty$ form a Rises basis, we prove the following theorem.

Theorem 3. *For each fixed* x *the equation (12) has a unique solution* $K(x,y)$ *in* $L_2(x,\infty)$.

Equation (12) is called the main equation of the inverse problem. The last theorems let solve inverse problem as follows. By means of spectral data $\{\lambda_n, \alpha_n > 0\}_{n=0}^{\infty}$ we build function $F(x,y)$ by formula (11). Solving the main equation (12) we find $K(x,y)$. Then the potential $q(x)$ is restored by (8).

REFERENCES

1. *Berezin F.A., Shubin M.* The Schrodinger Equation, Springer Science & Business Media, 1991, 555 p.
2. *Sakhnovich L. A.* Asymptotic behavior of the spectrum of an anharmonic oscillator, Theoretical and Mathematical Physics, 1981, v. 47, Issue 2, pp.449-456
3. *McKean H.P., Trubowitz E.* The spectral class of the quantum-mechanical harmonic oscillator, Comm. Math. Phys., **82**(1982), 471-495.
4. *Levitan B.M.* Sturm-Liouville operators on the entire real axis with the same discrete spectrum.Math. USSR-Sb. **60**(1), 77–106 (1988).
5. *Chelkak D., Kargaev P., Korotyaev E.* An Inverse Problem for an Harmonic Oscillator Perturbed by Potential: Uniqueness, *Lett. Math. Phys.* **64**:1 (2003), 7–21.
6. *Chelkak D. , Kargaev P., Korotyaev E.* Inverse Problem for Harmonic Oscillator Perturbed by Potential, Characterization, *Comm. Math. Phys.*, **249**:4(2004), 133–196.
7. *Chelkak D., Korotyaev E.* The inverse problem for perturbed harmonic oscillator on the half-line with a Dirichlet boundary condition, Annales Henri Poincare, 2007, v.8, Issue 6, pp. 1115-1150.
8. *Gasyimov M.G., Mustafaev B. A.* On the inverse problem of scattering theory for the an harmonic equation on a semi axis, (Russian) Soviet Math. Dokl., 17 (1976), 621–624.
9. *Bateman H.* Higher transcendental functions. Vol. II, New York-Toronto-London: McGraw-Hill, 1953.
10. *Abramowitz M. and Stegun I.* Handbook of mathematical functions, National Bureau of Standars Applied Mathematics Series,55, Issued June, 1964.

[1]Institute of Mathematics and Mechanics of ANAS

[2] Baku State University

agil_khanmamedov@yahoo.com

HƏYƏCANLANMIŞ HARMONİK OSSİLYATOR ÜÇÜN TƏRS SPEKTRAL MƏSƏLƏ

A.X.Xanməmmədov, H.M.Məsmalıyev, M.H.Mahmudova

Məqalədə anharmonik tənlik üçün tərs spektral məsələyə baxılmışdır. Çevirmə operatorunun köməyi və sonsuzluq şərti ilə məsələnin həlli yolu tapılmışdır. Tərs məsələnin əsas tənliyi alınmışdır. Tərs məsələnin həlli alqoritmi verilmişdir.

Açar sözlər: *həyəcanlanmış harmonik ossilyator, çevirmə operatoru, tərs məsələ, əsas tənlik*

ОБРАТНАЯ СПЕКТРАЛЬНАЯ ЗАДАЧА ДЛЯ ВОЗМУЩЕННОГО ГАРМОНИЧЕСКОГО ОСЦИЛЛЯТОРА

А.Х.Ханмамедов, Г.М.Масмалиев, М.Г.Махмудова

Рассмотрена обратная спектральная задача для ангармонического уравнения. С помощью оператора преобразования найдено представление решения этого уравнения с условием на бесконечности. Получено основное уравнение обратной задачи. Указан алгоритм решения обратной задачи.

Ключевые слова: *возмущенный гармонический осциллятор, оператор преобразования, обратная задача, основное уравнение*