

UDC 517.956.17

ON AN INVERSE PROBLEM FOR A "WEAK" SYSTEM OF PARABOLIC EQUATIONS

Corresponding Member of ANAS R.V.Huseynov, A.Ya.Akhundov

The goal of the paper is to study the well-posedness of an inverse problem on definition of unknown coefficients at higher derivatives in a "weak" system of parabolic equations.

A theorem on uniqueness of the solution of the considered inverse problem is proved and an estimation characterizing the stability of the solution is obtained.

Keywords: inverse problem, "weak" system of parabolic equations, uniqueness

Let R^n be n - dimensional Euclidean space of points $x = (x_1, \dots, x_n)$, $D \subset R^n$ be a bounded domain with smooth boundary ∂D ,

$\Omega = D \times (0, T]$, $S = \partial D \times [0, T]$, $0 < T$ - is a constant number.

The space

$$C^l(\cdot), C^{l+\alpha}(\cdot), C^{l, l/2}(\cdot), C^{l+\alpha, (l+\alpha)/2}(\cdot),$$

$l = 0, 1, 2$, $\alpha \in (0, 1)$ and the norms is these spaces were defined for example in [1, p.12-20],

$$u = (u_1, \dots, u_m),$$

$$\|u\|_l = \sum_{k=1}^m \|u_k\|_{C^l} = \sum_{k=1}^m \sum_{i=0}^l \sup_{\Omega} |D_x^i u_k|,$$

$$\|\psi(x, t, u)\|_0 = \sum_{k=1}^m \sup_{\Omega} |\psi_k(x, t, u(x, t))|,$$

$$u_{kt} = \frac{\partial u_k}{\partial t}, \quad u_{kx_j} = \frac{\partial u_k}{\partial x_j},$$

$$\frac{\partial u_k}{\partial \nu} = \sum_{j=1}^n \frac{\partial u_k}{\partial x_j} \cos(\nu, x_j), \quad \nu - \text{ is a unit}$$

vector of interval normal to S at its any point,

$$\Delta u_k = \sum_{i=1}^n \frac{\partial^2 u_k}{\partial x_i^2};$$

$$\int_D u_k(x, t) dx = \int_{(\bar{D})} \int u_k(x, t) dx_1 \dots dx_n,$$

$D_x^l u_k(x, t)$ are all possible derivatives of the function $u_k(x, t)$ with respect to x_j .

Consider an inverse problem on definition of $\{a_k(x), u_k(x, t), k = \overline{1, m}\}$ from the conditions

$$u_{kt} - a_k(x) \Delta u_k = f_k(x, t), \quad (x, t) \in \Omega, \quad (1)$$

$$u_k(x, 0) = \varphi_k(x), \quad x \in \bar{D} = D \cup \partial D, \quad (2)$$

$$\frac{\partial u_k}{\partial \nu} = \psi_k(x, t, u), \quad (x, t) \in S, \quad (3)$$

$$\int_0^T u_k(x, t) dt = h_k(x), \quad x \in \bar{D}, \quad (4)$$

where $f_k(x, t)$, $\varphi_k(x)$, $\psi_k(x, t, p)$, $h_k(x)$, $k = \overline{1, m}$ are the given functions.

Similar inverse problems were earlier considered in the papers [2,3,4] (see also references in these papers).

For initial data of problem (1)-(4) we make the following assumptions:

$$1^0. f_k(x, t) \in C^{\alpha, \alpha/2}(\bar{\Omega});$$

$$2^0. \varphi_k(x) \in C^{2+\alpha}(\bar{D});$$

3⁰. the functions $\psi = \psi_k(x, t, p), k = \overline{1, m}$ are determined and continuous in the totality of arguments in $A = \{(x, t, p) | (x, t) \in S, p \in R^m\}$;

for any $c_1 > 0$, $|p_k| \leq c_1$, the functions

$\psi_k(x, t, p)$ are Hölder continuous with respect to x and t with indices α and $\alpha/2$, respectively, for $(x, t) \in \bar{\Omega}$; there exists a $c_2 > 0$, such that for any $(x, t) \in \bar{\Omega}$ and $p, \bar{p} \in R^m$ the following inequality is fulfilled:

$$|\psi_k(x, t, p) - \psi_k(x, t, \bar{p})| \leq c_2 \sum_{k=1}^m |p_k - \bar{p}_k|;$$

4⁰. $h_k(x) \in C^{2+\alpha}(\bar{D})$, $h_1 T \leq \Delta h_k(x) \leq h_2 T$,
 $x \in \bar{D}$, $h_1, h_2 > 0$ - are constant numbers.

Definition 1. On pair of function $\{a_k(x), u_k(x, t), k = \overline{1, m}\}$ is said to be the solution of problem (1)-(4), if

- 1) $a_k(x) \geq c_4 > 0$, $a_k(x) \in C^\alpha(\bar{D})$;
- 2) $u_k(x, t) \in C^{2+\alpha, 1+\alpha/2}(\Omega) \cap C^{1+\alpha, \alpha/2}(\bar{\Omega})$;

3) for these functions relations (1)-(4) are fulfilled, and the boundary condition is determined in the following way:

$$\frac{\partial u(x, t)}{\partial \nu(x, t)} = \lim_{\substack{y \rightarrow x \\ y \in \sigma}} \frac{\partial u(y, t)}{\partial \nu(y, t)}$$

where σ is a closed cone with the vertex x , that is contained in $D \cup \{x\}$.

When considering inverse problems, the issues of uniqueness and stability of the solution occupies a special place.

Let $\{a_k^i(x), u_k^i(x, t), k = \overline{1, m}\}$ be the solutions of problem (1)-(4), corresponding to data $f_k^i(\cdot), \varphi_k^i(\cdot), \psi_k^i(\cdot), h_k^i(\cdot), i = 1, 2$.

Definition 2. We say that the solution of problem (1)-(4) is stable, if for any $\varepsilon > 0$ there will be found such as $\delta(\varepsilon) > 0$, that for $\|f^1 - f^2\|_0 \leq \delta$, $\|\varphi^1 - \varphi^2\|_2 \leq \delta$, $\|\psi^1 - \psi^2\|_0 \leq \delta$, $\|h^1 - h^2\|_2 \leq \delta$ the inequality $\|a^1 - a^2\|_0 + \|u^1 - u^2\|_0 \leq \varepsilon$ is fulfilled.

Theorem. Let 1) the functions $f_k^i(\cdot), \varphi_k^i(\cdot), \psi_k^i(\cdot), h_k^i(\cdot), k = \overline{1, m}, i = 1, 2$ satisfy conditions $1^0, 2^0, 3^0, 4^0$, respectively; 2) there exists the solutions $\{a_k^i(x), u_k^i(x, t), k = \overline{1, m}, i = 1, 2\}$ of problem (1)-(4) in the sense of definition 1 and they belong to the set

$$K_\alpha = \left\{ a_k, u_k \mid a_k(x) \in C^\alpha(\bar{D}), 0 < c_4 \leq a_k(x) \leq c_5, x \in \bar{D}, \right. \\ \left. u_k(x, t) \in C^{2+\alpha, 1+\alpha/2}(\bar{\Omega}), \right. \\ \left. |D_x^l u_k(x, t)| \leq c_6, l = 0, 1, 2, (x, t) \in \bar{\Omega}, \right. \\ \left. c_4, c_5, c_6 > 0 - \text{are constant numbers} \right\}.$$

Then there exists such $T^* (0 < T^* \leq T)$, that for $(x, t) \in \bar{D} \times [0, T^*]$, the solution of problem (1)-(4) is unique, and the following estimation is valid

$$\|a^1 - a^2\|_0 + \|u^1 - u^2\|_0 \leq \\ \leq c [\|f^1 - f^2\|_0 + \|\varphi^1 - \varphi^2\|_2 + \|\psi^1 - \psi^2\|_0 + \|h^1 - h^2\|_2], \quad (5)$$

where $c > 0$ is dependent on the data of problem (1)-(4) and the set K_α .

Proof. Integrating the both parts of equation (1) with respect to the variable t on the interval $(0, T)$ with regard to conditions of the theorem, for the functions $a_k(x)$ we get

$$a_k(x) = \left[u_k(x, T) - \varphi_k(x) - \int_0^T f_k(x, t) dt \right] (\Delta h_k(x))^{-1}, \quad x \in \bar{D}, \quad (6)$$

Assume

$$z_k(x, t) = u_k^1(x, t) - u_k^2(x, t), \quad \lambda_k(x) = a_k^1(x) - a_k^2(x),$$

$$\delta_{1k}(x, t) = f_k^1(x, t) - f_k^2(x, t), \\ \delta_{2k}(x) = \varphi_k^1(x) - \varphi_k^2(x), \\ \delta_{3k}(x, t, p) = \psi_k^1(x, t, p) - \psi_k^2(x, t, p), \\ \delta_{4k}(x) = h_k^1(x) - h_k^2(x), \quad k = \overline{1, m}.$$

It is easy to check that the functions $\lambda_k(x)$, $w_k(x, t) = z_k(x, t) - \delta_{2k}(x)$, $k = \overline{1, m}$ satisfy the following conditions:

$$w_{kt} - a_k^1(x) \Delta w_k = F_k(x, t), \quad (x, t) \in \Omega, \quad (7)$$

$$w_k(x, 0) = 0, \quad x \in \overline{D}, \quad (8)$$

$$\frac{\partial w_k}{\partial \nu} = \Phi_k(x, t), \quad (x, t) \in S, \quad (9)$$

$$\lambda_k(x) = z_k(x, T) (\Delta h_k^1)^{-1} - H_k(x), \quad x \in \overline{D}, \quad (10)$$

where

$$F_k(x, t) = \delta_{1k}(x, t) + \Delta \delta_{2k}(x) + \lambda_k(x) \Delta u_k^2(x, t),$$

$$\begin{aligned} \Phi_k(x, t) &= \delta_{3k}(x, t, u_1) - \frac{\partial \delta_{2k}}{\partial \nu} + \\ &+ \psi_{2k}(x, t, u_1) - \psi_{2k}(x, t, u_2), \\ H_k(x) &= \left\{ \left[\delta_{2k}(x) + \int_0^T \delta_{1k}(x, t) dt \right] \Delta h_k^2(x) + \right. \\ &+ \left[u_k^2(x, T) - \varphi_k^2(x) - \int_0^T f_k^2(x, t) dt \right] \Delta \delta_{4k}(x) \Big\} \times \\ &\times [\Delta h_k^1(x) \Delta h_k^2(x)]^{-1}. \end{aligned}$$

Under the theorem assumptions, the input data of the problem on definition of $w(x, t)$ from conditions (7), (8), (9) are Holderian, in other words, the conditions of the theorem are satisfied [1, 334]. So, there exists a classic solution of problem (7), (8), (9) and it can be represented in the form [5, 182].

$$\begin{aligned} w_k(x, t) &= \int_0^t \int_D \Gamma_k(x, t; \xi, \tau) F_k(\xi, \tau) d\xi d\tau + \\ &+ \int_0^t \int_{\partial D} \Gamma_k(x, t; \xi, \tau) \rho_k(\xi, \tau) d\xi_0 d\tau, \end{aligned} \quad (11)$$

here $\Gamma_k(x, t; \xi, \tau)$ is fundamental solution of the equations $w_{kt} - a_k^1(x) \Delta w_k = 0$, $k = \overline{1, m}$, $d\xi = d\xi_1 \dots d\xi_n$, $d\xi_0$ is an element of the surface ∂D , $\rho_k(x, t)$ is a bounded, continuous solution of the integral equation [5, 182].

$$\begin{aligned} \rho_k(x, t) &= 2 \int_0^t \int_D \frac{\partial \Gamma_k(x, t; \xi, \tau)}{\partial \nu} F_k(\xi, \tau) d\xi d\tau + \\ &+ 2 \int_0^t \int_{\partial D} \frac{\partial \Gamma_k(x, t; \xi, \tau)}{\partial \nu} \rho_k(\xi, \tau) d\xi_0 d\tau - 2 \Phi_k(x, t) \end{aligned} \quad (12)$$

Denote

$$\chi = \|u^1 - u^2\|_0 + \|a^1 - a^2\|_0.$$

Estimate the function $|z_k(x, t)|$, $k = \overline{1, m}$. Taking into account $z_k(x, t) = w_k(x, t) + \delta_{2k}(x)$, from (11) we get:

$$\begin{aligned} |z_k(x, t)| &\leq |\delta_{2k}(x)| + \int_0^t \int_D \Gamma_k(x, t; \xi, \tau) |F_k(\xi, \tau)| d\xi d\tau + \\ &+ \int_0^t \int_{\partial D} \Gamma_k(x, t; \xi, \tau) |\rho_k(\xi, \tau)| d\xi_0 d\tau. \end{aligned} \quad (13)$$

For the first summand in the right side of (13), by definition we have:

$$|\delta_{2k}(x)| = |\varphi_k^1(x) - \varphi_k^2(x)| \leq \|\varphi^1 - \varphi^2\|_0, \quad x \in \overline{D}. \quad (14)$$

For the fundamental solutions $\Gamma_k(x, t; \xi, \tau)$, $k = \overline{1, m}$ the following estimations are valid [1, 444].

$$\int_{R^n} \Gamma_k(x, t; \xi, \tau) d\xi \leq c_7, \quad (15)$$

$$\left| \int_{R^n} D_x^l \Gamma_k(x, t; \xi, \tau) d\xi \right| \leq c_8 (t - \tau)^{\frac{l-\alpha}{2}}, \quad l = 1, 2 \quad (16)$$

The expression $\int_0^t \int_D \Gamma_k(x, t; \xi, \tau) d\xi d\tau$, appearing in the second summand of the right side of (13), is estimated with inequality (15) taken into account:

$$\int_0^t \int_D \Gamma_k(x, t; \xi, \tau) d\xi d\tau \leq \int_0^t d\tau \int_{R^n} \Gamma_k(x, t; \xi, \tau) d\xi \leq c_9 t \quad (17)$$

The integrand function $|F_k(x, t)|$, $k = \overline{1, m}$ in the second summand of the right hand side of (13), with regard to conditions imposed on initial data and on the set K_α , satisfies the estimation

$$\begin{aligned} |F_k(x, t)| &\leq |\delta_{1k}(x, t)| + |\Delta \delta_{2k}(x)| + |\lambda_k(x)| |\Delta u_k^2| \leq \\ &\leq \|f^1 - f^2\|_0 + \|\varphi^1 - \varphi^2\|_2 + c_{10} \chi, \\ &(x, t) \in \overline{\Omega}, \end{aligned} \quad (18)$$

where $c_{10} > 0$ is dependent on the data of problem (1)-(4) and the set K_α .

The expression $\int_0^t \int_{\partial D} \Gamma_k(x, t; \xi, \tau) d\xi d\tau$, in the their summand of the right hand side of (13), is estimated subject to inequality (16) and the Gauss-Ostrogradskii formula:

$$\int_0^t \int_{\partial D} \Gamma_k(x, t; \xi, \tau) d\xi d\tau \leq \int_0^t d\tau \int_D \left| D_x^1 \Gamma_k(x, t; \xi, \tau) \right| d\xi \leq \int_0^t d\tau \int_{R^n} \left| D_x^l \Gamma_k(x, t; \xi, \tau) \right| d\xi \leq c_{11} t^{\frac{1+\alpha}{2}}, \quad (19)$$

The integrand function $|\rho_k(x, t)|$, $k = \overline{1, m}$ with regard to the theorem conditions, the set K_α and inequality (19), for $l=1$ и $l=2$ is estimated from (12):

$$|\rho_k(x, t)| \leq c_{12} [\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \chi] + c_{13} \|\rho\|_0 t^{\alpha/2}, \quad (x, t) \in S$$

where $c_{12}, c_{13} > 0$ is dependent on the data of problem (1)-(4) and the set K_α .

The last inequality is satisfied for all $(x, t) \in \partial D \times [0, T]$, therefore it should be satisfied maximum values of the left hand side as well:

$$\|\rho\|_0 \leq c_{12} [\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \chi] + c_{13} t^{\alpha/2} \|\rho\|_0.$$

Let $0 < T_1 \leq T$ be a number such that $c_{13} T_1^{\alpha/2} < 1$. Then for all $(x, t) \in \partial D \times [0, T_1]$ the following estimation is valid:

$$\|\rho\|_0 \leq c_{14} [\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \chi], \quad (20)$$

where $c_{14} > 0$ is dependent on the data of problem (1)-(4) and the set K_α .

Taking into account inequalities (14), (17), (18), (19), (20) from (13) we get:

$$|z_k(x, t)| \leq c_{15} [\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0] + c_{16} t^\alpha \chi, \quad (x, t) \in \overline{\Omega} \quad (21)$$

where $c_{15}, c_{16} > 0$ is dependent on the data of problem (1)-(4) and the set K_α .

Now estimate the function $|\lambda_k(x)|$, $k = \overline{1, m}$. From formula (10) we have:

$$|\lambda_k(x)| \leq |z_k(x, T)| |h_k^1(x)^{-1}| + |H_k(x)|.$$

By the condition imposed on initial data and on the set K_α , taking into account inequality (21) and behaving in the same way as in deriving inequality (21), for $|\lambda_k(x)|$ for the last inequality we get:

$$|\lambda_k(x)| \leq c_{17} [\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \|\delta_4\|_2] + c_{18} t^\alpha \chi, \quad x \in \overline{D} \quad (22)$$

where $c_{17}, c_{18} > 0$ is dependent on the data of problem (1)-(4) and the set K_α .

Inequalities (21) and (22) are satisfied for all $(x, t) \in \overline{D} \times [0, T_1]$. So, they should be satisfied for maximum values of the left sides as well. Combining the obtained inequality, we have:

$$\chi \leq c_{19} [\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \|\delta_4\|_2] + c_{20} t^{\alpha/2} \chi, \quad (23)$$

where $c_{19}, c_{20} > 0$ is dependent on the data of problem (1)-(4) and the set K_α .

Let T_2 ($0 < T_2 \leq T$) be a number such that $c_{20} T_2^{\alpha/2} < 1$. Then from (23) we get $(x, t) \in \overline{D} \times [0, T^*]$ $T^* = \min\{T_1, T_2\}$. The following estimation is valid

$$\begin{aligned} \|u^1 - u^2\|_0 + \|a^1 - a^2\|_0 &\leq c_{21} \|f^1 - f^2\|_0 + \\ &+ \|\varphi^1 - \varphi^2\|_2 + \|\psi^1 - \psi^2\|_0 + \|h^1 - h^2\|_2 \end{aligned}$$

where $c_{21} > 0$ is dependent on the data of problem (1)-(4) and the set K_α .

The uniqueness of the solution of problem (1)-(4) follows from the last estimation for

$f_k^1(x, t) = f_k^2(x, t)$, $\varphi_k^1(x) = \varphi_k^2(x)$, $\psi_k^1(x, t, p) = \psi_k^2(x, t, p)$, $h_k^1(x) = h_k^2(x)$. The theorem is proved.

The work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan Grant №EIF-2013-9(15)-46/12/1

REFERENCES

1. Ladyzhenskaya O.A., Solonnikov V.A., Ural'tseva N.I. Linear and quasilinear equations of parabolic type. M:Nauka, 1967, 736p.

2. Akhundov A.Y. Some inverse problems for strong parabolic systems //Ukrainian Mathematical Journal, 2006, v.58, №1, p.114-123.

3. Akhundov A.Y., Gasanova A.I. Determination of the coefficient of a semilinear parabolic equation in the case of boundary-value problem with nonlinear boundary condition// Ukrainian Mathematical Journal, 2014, v.66, №6, p.847-852

4. Iskenderov A.D., Akhundov A.Ya. Inverse problem for a linear system of parabolic equations// DAN SSSR, 2009, vol.424, №4, p.442-444.

5. Fridman A. Parabolic type partial equations. M., 1968, 427p.

*Institute of Mathematics and Mechanics
of ANAS
adalatakhund@mail.ru*

"ZƏİF" PARABOLİK TƏNLİKLƏR SİSTEMİ ÜÇÜN BİR TƏRS MƏSƏLƏ HAQQINDA

R.V.Hüseynov, Ə.Y.Axundov

Məqalədə "zəif" parabolik tənliklər sistemində yüksək tərtibli törəmələr qarşısında duran naməlum əmsalların tapılması haqqında tərs məsələnin korrektiliyi öyrənilir. Həllin yeganəliyi, "şərti" dayanaqlığı haqqında teorem isbat olunmuşdur.

Açar sözlər: tərs məsələ, "zəif" parabolik tənliklər sistemi, yeganəlik

ОБ ОДНОЙ ОБРАТНОЙ ЗАДАЧЕ ДЛЯ «СЛАБОЙ» СИСТЕМЫ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ

Р.В.Гусейнов, А.Я.Ахундов

Целью настоящей работы является исследование корректности обратной задачи об определении неизвестных коэффициентов при старших производных в «слабой» системе параболических уравнений.

Доказана теорема о единственности решения рассматриваемой обратной задачи и получена оценка, характеризующая устойчивость решения.

Ключевые слова: обратная задача, "слабая" система параболических уравнений, единственность