

UDC 517.98

ON ISOMORPHISM OF TWO BASES IN MORREY-LEBESGUE TYPE SPACES

Corresponding Member of ANAS B.T.Bilalov¹, R.H.Abdullayeva²

Double system of exponents with complex-valued coefficients is considered. Under some conditions on the coefficients, we prove that if this system forms a basis for the Morrey-Lebesgue type space on $[-\pi, \pi]$, then it is isomorphic to the classical system of exponents in this space.

Keywords: Morrey-Lebesgue type space, system of exponents, isomorphism, basicity

Introduction

Consider the double system of exponents

$$\{A(t)e^{int}; B(t)e^{-ikt}\}_{n \in Z_+, k \in N}, \quad (1)$$

with complex-valued coefficients $A(t) = |A(t)|e^{i\alpha(t)}$, $B(t) = |B(t)|e^{i\beta(t)}$ on the interval $[-\pi, \pi]$, where N is the set of natural numbers and $Z_+ = \{0\} \cup N$. Criterion for the basicity of a system of exponents:

$$\{e^{i(n+\alpha \operatorname{sign} n)t}\}_{n \in Z}, \quad (2)$$

for $L_p(-\pi, \pi)$ $1 < p < +\infty$, where Z is the set of all integers, was first found by A.M. Sedletsky [1].

Some questions of the approximation theory were studied in [1-5;9].

In this paper, we consider a system of exponents (1) and prove that if it forms a basis for Morrey-Lebesgue type space $M_{p,\alpha} = M_{p,\alpha}(-\pi, \pi)$, then it is isomorphic to the classical system of exponents in this space. To do so, we use the method of boundary value problems. A similar result was obtained earlier in [8].

Necessary information

We will need some facts about the theory of Morrey-type spaces. Let Γ be a rectifiable Jordan curve in the complex plane C . By $|M|_\Gamma$ we denote the linear Lebesgue measure of the set $M \subset \Gamma$.

By Morrey-Lebesgue space $L^{p,\alpha}(\Gamma)$,

$0 \leq \alpha \leq 1$, $p \geq 1$, we mean the normed space of all measurable functions $f(\cdot)$ on Γ with the finite norm $\|\cdot\|_{L^{p,\alpha}(\Gamma)}$:

$$\|f\|_{L^{p,\alpha}(\Gamma)} = \sup_B \left(|B \cap \Gamma|^{\alpha-1} \int_{B \cap \Gamma} |f(\xi)|^p |d\xi| \right)^{1/p} < +\infty.$$

$L^{p,\alpha}(\Gamma)$ is the Banach space with $L^{p,1}(\Gamma) = L_p(\Gamma)$, $L^{p,0}(\Gamma) = L_\infty(\Gamma)$. For $\Gamma = [-\pi, \pi]$ we will use the notation $L^{p,\alpha}(-\pi, \pi) \subset L^{p,\alpha}$. More details on Morrey-type spaces can be found in [6-15].

Let $\omega = \{z \in C : |z| < 1\}$ be the unit circle on C and $\partial\omega = \gamma$ be its circumference. Define the Morrey-Hardy space $H_+^{p,\alpha}$ of analytic functions $f(z)$ inside ω equipped with the following norm:

$$\|f\|_{H_+^{p,\alpha}} = \sup_{0 < r < 1} \|f(re^n)\|_{L^{p,\alpha}}.$$

Denote by $\tilde{L}^{p,\alpha}$ the linear subspace of $L^{p,\alpha}$ functions, whose shifts are continuous in $L^{p,\alpha}$, i.e. $\|f(\cdot + \delta) - f(\cdot)\|_{L^{p,\alpha}} \rightarrow 0$ as $\delta \rightarrow 0$. Consider the closure of $\tilde{L}^{p,\alpha}$ in $L^{p,\alpha}$ and denote it by $M^{p,\alpha}$.

In the sequel, we will extensively use the following lemma.

Lemma 1. If $f \in L_\infty \wedge g \in M^{p,\alpha}$,

$1 < p < +\infty$, $0 < \alpha \leq 1$, then $fg \in M^{p,\alpha}$.

Consider the following singular operator:

$$(Sf)(\tau) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi) d\xi}{\xi - \tau}, \quad \tau \in \gamma.$$

Using the results of [7;11;14], it is easy to prove the following

Theorem 1. *Singular operator S acts boundedly in $M^{p,\alpha}(\gamma)$ when $0 < \alpha \leq 1$ and $1 < p < +\infty$.*

Consider the space $H_+^{p,\alpha}$. Denote by $L_+^{p,\alpha}$ the subspace of $L^{p,\alpha}$, generated by the restrictions of the functions from $H_+^{p,\alpha}$ to γ . From the results mentioned above it immediately follows that the spaces $H_+^{p,\alpha}$ and $L_+^{p,\alpha}$ are isomorphic and $f^+(\cdot) = (Jf)(\cdot)$, where $f \in H_+^{p,\alpha}$, f^+ are nontangential boundary values of f on γ , and J performs the corresponding isomorphism. Let $M_+^{p,\alpha} = M^{p,\alpha} \cap L_+^{p,\alpha}$. It is clear that $M_+^{p,\alpha}$ is a subspace of $M^{p,\alpha}$ with respect to the norm $\|\cdot\|_{L^{p,\alpha}}$. Let $MH_+^{p,\alpha} = J^{-1}(M_+^{p,\alpha})$. The latter is a subspace of $H_+^{p,\alpha}$. Let $f \in H_+^{p,\alpha}$ and f^+ be its boundary values. It is absolutely clear that the norm in $\|f\|_{H_+^{p,\alpha}}$ can be defined also as $\|f\|_{H_+^{p,\alpha}} = \|f^+\|_{L^{p,\alpha}}$.

Similar to the classical case, we define the Morrey-Hardy class outside ω . Let $D^- = C \setminus \bar{\omega}$ ($\bar{\omega} = \omega \cup \gamma$). We say the analytic function f in D^- has finite order k at infinity, if its Laurent series in the neighborhood of the point at infinity has the following form:

$$f(z) = \sum_{n=-\infty}^k a_n z^n, \quad k < +\infty, \quad a_k \neq 0.$$

Thus, for $k > 0$ the function $f(z)$ has a pole of order k ; for $k = 0$ it is bounded; and in case of $k < 0$ it has a zero of order $(-k)$. Let $f(z) = f_0(z) + f_1(z)$, where $f_0(z)$ is the principal, and $f_1(z)$ is the regular part of decomposi-

tion (1) of the function $f(z)$. Consequently, if $k \leq 0$, then $f_0(z) = 0$. If $k > 0$, then $f_0(z)$ is a polynomial of degree k . We say that the function $f(z)$ belongs to the class ${}_m H_-^{p,\alpha}$, if it has the order at infinity less or equal to m , i.e. $k \leq m$ and $f_1\left(\frac{1}{z}\right) \in H_+^{p,\alpha}$.

The class ${}_m MH_-^{p,\alpha}$ is defined in a very similar manner to the case of $MH_+^{p,\alpha}$. In other words, ${}_m MH_-^{p,\alpha}$ is the subspace of ${}_m H_-^{p,\alpha}$ - functions whose shifts are continuous on the unit circle's circumference with respect to the norm $\|\cdot\|_{L^{p,\alpha}(\gamma)}$.

We will need the Sokhotski-Plemelj formula for the boundary values of a Cauchy type integral

$$\Phi(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\tau) d\tau}{\tau - z},$$

where $f(e^{it}) \in L_1(-\pi, \pi)$. Then the boundary values $\Phi^\pm(\tau)$, $\tau \in \gamma$, satisfy the following Sokhotski-Plemelj expression:

$$\Phi^\pm(\tau) = \pm \frac{1}{2} f(\tau) + (Sf)(\tau), \quad a.e. \quad \tau \in \gamma. \quad (3)$$

From the Sokhotski-Plemelj formulas (5) it follows that

$$f(\tau) = \Phi^+(\tau) - \Phi^-(\tau), \quad a.e. \quad \tau \in \gamma. \quad (4)$$

Applying Theorem 1 to the expression (4), we obtain in a similar way that the direct decomposition

$$M^{p,\alpha} = MH_+^{p,\alpha} \dot{+} {}_{-1}MH_-^{p,\alpha} \quad (5)$$

also holds.

Let $P^\pm$ be projectors

$P^+ : M^{p,\alpha} \rightarrow M_+^{p,\alpha} \wedge P^- : M^{p,\alpha} \rightarrow {}_{-1}M_-^{p,\alpha}$ generated by the decomposition (5) (or (11)). Denote by $T^\pm : M^{p,\alpha} \rightarrow M^{p,\alpha}$ the multiplication operators defined as follows:

$$T^+ f = Af \wedge T^- f = Bf, \quad \forall f \in L^{p,\alpha}.$$

Consider the operator $T = T^+ P^+ + T^- P^-$.

Thus, we have proved the following theorem:

Theorem 2. Let $A^{\pm 1}; B^{\pm 1} \in L_{\infty}(-\pi, \pi)$.

The system (1) forms a basis for $M^{p,\alpha}$ only when the equation $Tf = g$, $g \in M^{p,\alpha}$ is correctly solvable in $M^{p,\alpha}$.

Now let's prove the following main theorem.

Theorem 3. Let $A^{\pm 1}; B^{\pm 1} \in L_{\infty}(-\pi, \pi)$. If the system (1) forms a basis for $M^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha \leq 1$, then it is isomorphic to the classical system of exponents $\{e^{\text{int}}\}_{n \in \mathbb{Z}}$ in $M^{p,\alpha}$, with the isomorphism given by means of the operator T_0 :

$$(T_0 f)(t) = A(t) \sum_{n=0}^{\infty} (f; e^{inx}) e^{\text{int}} + \\ B(t) \sum_{n=1}^{\infty} (f; e^{-inx}) e^{-\text{int}},$$

where

$$(f; g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \overline{g(t)} dt.$$

This theorem has the following immediate corollary:

Corollary 1. If the perturbed system of exponents

$$\left\{ e^{i(n+\alpha \text{sign} n)t} \right\}_{n \in \mathbb{Z}}$$

forms a basis for the space $M^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha \leq 1$, then it is isomorphic to the classical system of exponents $\{e^{\text{int}}\}_{n \in \mathbb{Z}}$ in this space.

This work was supported by the Research Program Competition launched by the National

Academy of Sciences of Azerbaijan (Program: Frame theory Applications of Wavelet Analysis to Signal Processing in Seismology and Other Fields).

REFERENCES

1. Мусеев Е.И. О базисности систем синусов и косинусов// ДАН СССР, 1984, т. 275, №4, с. 794-798.
2. Билалов Б.Т. О равномерной сходимости рядов по одной системе синусов// Дифф. уравнения, 1988, т.24, №1, с. 175-177.
3. Билалов Б.Т. Базисность некоторых систем функций// Дифф. уравнения, 1989, т. 25, с.163-164.
4. Билалов Б.Т. Базисность некоторых систем экспонент, косинусов и синусов// Дифф. уравнения. 1990, т.26, №1, с. 10-16.
5. Билалов Б.Т. Базисные свойства некоторых систем экспонент, косинусов и синусов// Сибирский матем. журнал, 2004, Т.45, №2, с.264-273.
6. Israfilov D. M. and Tozman N. P. Approximation by polynomials in Morrey–Smirnov classes// East J. Approx. 14(3) (2008) 255–269.
7. Israfilov D. M. and Tozman N. P., Approximation in Morrey–Smirnov classes// Azerbaijan J. Math. 1(1) (2011) 99–113.
8. Bilalov B.T., Quliyeva A.A. On basicity of exponential systems in Morrey-type spaces// International Journal of Mathematics. Vol. 25, No. 6 (2014) 1450054 (10 pages).
9. Bilalov B.T. On isomorphism of two bases in L_p , Fundam. Prikl. Mat., 1:4 (1995), 1091–1094.
10. Chen Y., Regularity of the solution to the Dirichlet problem in Morrey space// J. Partial Differ. Eqs. 15 (2002) 37–46.
11. Kokilashvili V. and Meskhi A., Boundedness of maximal and singular operators in Morrey spaces with variable exponent, Govern. College Univ. Lahore 72 (2008) 1–11.
12. Mazzucato A. L., Decomposition of Besov-Morrey spaces, in "Harmonic Analysis at Mount Holyoke", American Mathematical Society Contemporary Mathematics, 320(2003) 279-294.
13. Peetre J., On the theory of spaces// J. Funct. Anal. 4 (1964) 71-87.
14. Samko N., Weight Hardy and singular operators in Morrey spaces// J. Math. Anal. Appl. 35(1) (2009) 183–188.
15. C. T. Zorko, Morrey space, Proc. Amer. Math. Soc. 98(4) (1986) 586–592.

¹*Institute of Mathematics and Mechanics of ANAS*

²*School №297, Azerbaijan, Baku*
b_bilalov@mail.ru

MORRI-LEBEQ TİP FƏZALARDA İKİ BAZİSİN İZOMORFLUĞU HAQQINDA**B.T.Bilalov, R.H.Abdullayeva**

Kompleks qiymətli əmsala malik ikiqat eksponent sisteminə baxılır. Əmsal üzərinə müəyyən şərtlər qoymaqla isbat edilmişdir ki, əgər bu sistem Morri-Lebeq tip fəzada $[-\pi, \pi]$ aralığında bazis təşkil edirsə, onda bu sistem bu fəzada klassik eksponent sisteminə izomorfdur.

Açar sözlər: Morri-Lebeq tip fəza, eksponent sistemi, izomorfizm, bazislik

**ОБ ИЗОМОРФИЗМЕ ДВУХ БАЗИСОВ В ПРОСТРАНСТВАХ
ТИПА МОРРИ-ЛЕБЕГ****Б.Т.Билалов, Р.Г.Абдуллаева**

Рассматривается двойная система экспонент с комплекснозначными коэффициентами. При определенных условиях на коэффициенты доказывается, что если эта система образует базис в пространстве типа Морри-Лебег на промежутке $[-\pi, \pi]$, то она изоморфна к классической системе экспонент в этом пространстве.

Ключевые слова: пространство типа Морри-Лебег, система экспонент, изоморфизм, базисность