

ON BASES IN BANACH SPACES

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A direct expansion of a Banach space with respect to subspaces is considered. A method for constructing a basis for a space proceeding from bases for subspaces is offered. We also consider the cases when the bases for subspaces are isomorphic and the corresponding isomorphisms may not hold. And we study the completeness and the minimality of corresponding systems.

Keywords: basis, completeness, minimality

Introduction

Study of the spectral properties of many discrete differential operators requires new methods for constructing or establishing bases. In case of discontinuous differential operator, there arise the systems of eigenfunctions that cannot be treated for basicity by the earlier methods. To shed some light on this situation, we consider the following model spectral problem for second order discontinuous differential operator

$$-y''(x) = \lambda y(x), \quad x \in (-1, 0) \cup (0, 1),$$

with the boundary conditions

$$\begin{aligned} y(-1) = y(1) = 0; \quad y(-0) = y(+0), \\ y'(-0) - y'(0) = \lambda m y(0). \end{aligned}$$

This spectral problem has two sets of eigenfunctions [1]:

$$u_{1n}(x) = \sin \pi n x, \quad x \in [-1, 1], \quad n \in N,$$

and

$$u_{2n}(x) = \begin{cases} \sin \pi n x + o\left(\frac{1}{n}\right), & x \in [-1, 0], \\ -\sin \pi n x + o\left(\frac{1}{n}\right), & x \in [0, 1], \quad n \in N. \end{cases}$$

Such spectral problems arise when solving the problem of a loaded string fixed at both

ends with a load placed in the middle of the string by the Fourier method [2;3]. The use of this method requires the study of basis properties of the double system $\{u_{1n}; u_{2n}\}_{n \in N}$ in corresponding spaces of functions (usually in the Lebesgue or Sobolev spaces). Of course, it should be started with the basis properties of the system $\{u_{1n}; \mathcal{G}_n\}_{n \in N}$, which is the principal part of the asymptotics of the system $\{u_{1n}; u_{2n}\}_{n \in N}$:

$$\mathcal{G}_n(x) = \begin{cases} \sin \pi n x, & x \in [-1, 0], \\ -\sin \pi n x, & x \in [0, 1] \end{cases}$$

This is usually followed by the application of various perturbation methods. This approach is well studied (see, e.g., the articles [5-10] and the monographs [4; 11-15]. On the other hand, it is not difficult to see that the principal part $\{u_{1n}; \mathcal{G}_n\}_{n \in N}$ is not a standard (in other words, classical) system. It turns out that the form of the system $\{u_{1n}; \mathcal{G}_n\}_{n \in N}$ is not special, i.e. it can be derived from the general case. The general approach to these systems allows introducing a new method for constructing bases with a lot of applications in the spectral theory of differential operators.

In this work, we consider an abstract approach to the above problem. We consider a direct expansion of a Banach space with respect to subspaces. We offer a method for constructing a basis for a space proceeding from bases for subspaces. We also consider the cases when

the bases for subspaces are isomorphic and the corresponding isomorphisms may not hold. And we study the completeness, the minimality and the uniform minimality of corresponding systems.

Completeness and minimality

In the sequel, we will use the following construction and some obvious facts. Let the following direct sum hold

$$X = X_1 \oplus \dots \oplus X_m,$$

where X_i , $i = \overline{1, m}$, are some B -space. For convenience, we will represent the elements of the space X in the form of a vector

$$x \in X \Leftrightarrow x = (x_1, x_2, \dots, x_m),$$

where $x_k \in X_k$, $k = \overline{1, m}$. The norm in X will be defined by the formula

$$\|x\|_X = \sqrt{\sum_{i=1}^m \|x_i\|_{X_i}^2}.$$

Suppose we are given some system $\{u_{in}\}_{n \in N} \subset X_i$ for every $i = \overline{1, m}$. Consider the following system in the space X :

$$u_{in}^0 = \left(\underbrace{0, \dots, u_{in}, \dots, 0}_i \right), \quad i = \overline{1, m}; \quad n \in N.$$

Consider the following system in the space X :

$$\omega_{in} = (a_{i1}u_{1n}; \dots; a_{im}u_{mn}), \quad i = \overline{1, m}; \quad n \in N, \quad (1)$$

where a_{ij} are some numbers. Let

$$A = (a_{ij})_{i,j=\overline{1,m}}; \quad \Delta = \det A.$$

The following theorem is true.

Theorem 1. Let the system $\{u_{in}\}_{n \in N}$ be

complete (minimal) in the space X_i , $i = \overline{1, m}$. If $\Delta \neq 0$, then the system $\{\omega_{in}\}_{i=\overline{1,m}, n \in N}$ is also complete (minimal) in the space X .

In case $\Delta = 0$ we have the following theorem.

Theorem 2. Let the system $\{u_{in}\}_{n \in N}$ be minimal in X_i , for every $i = \overline{1, m}$. If $\Delta = 0$, then the system $\{\omega_{in}\}_{i=\overline{1,m}, n \in N}$, defined by (1), is not minimal in X .

Proof. Let's show that the system $\{\omega_{in}\}_{i=\overline{1,m}}$ is linearly dependent for every $n \in N$. From $\det(a_{ij}) = 0$ it follows that there exist the numbers c_i , $i = \overline{1, m}$, not all equal to zero, such that

$$\sum_{i=1}^m a_{ij}c_i = 0, \quad j = \overline{1, m}.$$

Then

$$\begin{aligned} \sum_{i=1}^m c_i \omega_{in} &= \sum_{i=1}^m c_i \sum_{j=1}^m a_{ij} u_{jn}^0 = \\ &= \sum_{j=1}^m \left(\sum_{i=1}^m a_{ij} c_i \right) u_{jn}^0 = 0, \quad \forall n \in N. \end{aligned}$$

Thus, in case $\Delta = 0$ the system $\{u_{in}\}_{i=\overline{1,m}, n \in N}$ is linearly dependent and not minimal. Theorem is proved.

The following theorem is also true.

Theorem 3. Let the system $\{u_{in}\}_{n \in N}$ be complete and minimal in X_i , for every $i = \overline{1, m}$. If $\Delta = 0$ then the system $\{\omega_{in}\}_{i=\overline{1,m}, n \in N}$ is not complete and not minimal in X .

Basicity

Suppose a direct expansion $X = X_1 \oplus \dots \oplus X_m$ holds, where X_k , $k = \overline{1, m}$ are some B -space. Let $T_{ij} : X_i \rightarrow X_j$ be some operators. Consider the system

$$\sum_{i=1}^m a_{ij} T_{ij} x_i = y_j, \quad j = \overline{1, m}, \quad (2)$$

where $y_j \in X_j$, $j = \overline{1, m}$ are the given, and $x_i \in X_i$, $i = \overline{1, m}$ are the sought elements. Assume that the spaces X_k , $k = \overline{1, m}$, are pairwise isomorphic and T_{ij} performs a corresponding isomorphism. Besides, assume that the following conditions are satisfied:

$$A) T_{ii} = I_i, \quad T_{ij} = T_{ji}^{-1},$$

$T_{jk} T_{ij} = T_{ik}$, $\forall i, j = \overline{1, m}$, where I_i is the identity operator in X_i .

Thus, the following lemma is true.

Lemma 1. *Let the operators $T_{ij} : X_i \rightarrow X_j$ perform an isomorphism between X_i and X_j , the conditions A) be satisfied and $\Delta \neq 0$. Then the system (2) is uniquely solvable for $\forall y \in X$, $y = (y_1, \dots, y_m)^t$ and, moreover, $\exists M > 0$*

$$\|x\|_X \leq M \|y\|_X, \quad (3)$$

where $x = (x_1, \dots, x_m)$.

The following theorem is true.

Theorem 4. *Let $T_{ij} \in L(X_i, X_j)$ be an isomorphism, the conditions A) be satisfied and $\Delta \neq 0$. Then the operator $T : X \rightarrow X$, defined by the matrix $(a_{ij} T_{ij})_{i,j=1}^m$, is an automorphism in $X = X_1 \oplus \dots \oplus X_m$.*

Suppose that the systems $\{u_{in}\}_{n \in N}$ form bases for the spaces X_i , $i = \overline{1, m}$, respectively. Let these bases be isomorphic and $T_{ij} \in L(X_i, X_j)$ perform a corresponding isomorphism: $T_{ij} u_{in} = u_{jn}$, $n \in N$. It is clear that the system $u_{in}^0 = (0, \dots, u_{in}, \dots, 0)$, $i = \overline{1, m}$; $n \in N$, forms a basis for X . Then, obviously, the system $\omega_{in} = T u_{in}^0 = (a_{i1} u_{1n}, a_{i2} u_{2n}, \dots, a_{im} u_{mn})$, also forms a basis for X . So the following theorem is true.

Theorem 5. *Let the B -spaces X_i , $i = \overline{1, m}$, be pairwise isomorphic and the systems $\{u_{in}\}_{n \in N}$ form isomorphic bases for them, respectively. If $\Delta \neq 0$, then the system $\{\omega_{in}\}_{i=\overline{1, m}; n \in N}$ forms a basis for X , isomorphic to the basis $\{u_{in}^0\}_{i=\overline{1, m}; n \in N}$.*

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BANAX FƏZALARINDA BAZİSLƏR HAQQINDA**B.T.Bilalov, T.B.Qasimov**

Banax fəzasının alt fəzalarının düz cəminə ayrılışına baxılır. Alt fəzaların bazisindən çıxış edərək fəzanın bazisinin alınmasının bir üsulu təklif olunur. Alt fəzaların izomorf olduğu və bu izomorfizmin tələb olunmadığı hallara ayrılıqda baxılır. Həmçinin müvafiq sistemlərin tamlıq və minimallığına baxılır.

Açar sözlər: bazis, tamlıq, minimallıq

О БАЗИСАХ В БАНАХОВЫХ ПРОСТРАНСТВАХ**Б.Т.Билалов, Т.Б.Касумов**

Рассматривается прямое разложение банахова пространства по подпространствам. Предлагается один способ образования базиса пространства исходя из базисов подпространств. Отдельно рассматриваются случаи, когда базисы подпространств изоморфны, и когда соответствующие изоморфизмы могут не иметь место. Также рассматриваются полнота и минимальность соответствующих систем.

Ключевые слова: базис, полнота, минимальность