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Mathematics**CONTROL OF THE MOVING SOURCES FOR WAVE EQUATION****R.A.Teymurov**

(Presented by Academician of NAS of Azerbaijan A.D.Gadjiev)

The problem of optimal control of the moving sources for oscillatory system whose state are described by the wave equation is investigated in this paper. The necessary conditions for optimality were established in the form of the pointwise and integral principles of maximum.

Keywords: mobile sources, integral identity, Hamilton-Pontryagin function, necessary optimality conditions, principle of a maximum

1. Introduction

The problems of optimal control of moving sources have numerous applications and great applied importance [1-5]. In these works problems of boundary control of processes of the oscillations described by the wave equation were investigated. In view of complexity of the solution of a problem of control of moving sources whose state are described by the partial differential equations, were studied insufficiently [6-9].

The variation method of the solution of a problem of optimal control of the moving sources for oscillatory system whose state are described by the wave equation is considered in the present paper. For the problem of optimal control considered below the theorem of existence and uniqueness of the solution is proved, necessary conditions for optimality in the form of the pointwise and integrated principles of maximum are received, sufficient conditions of Frechet differentiability of the performance criterion and expression for its gradient were determined.

2. Problem statement

Let's consider the propagation of one-dimensional waves in a string described by the wave equation

$$u_{tt} = a^2 u_{xx} + \sum_{k=1}^n p_k(t) \delta(x - s_k(t)), \quad (x, t) \in \Omega, \quad (1)$$

with initial and boundary conditions

$$u_x(0, t) = 0, \quad u_x(l, t) = 0, \quad 0 < t \leq T, \quad (2)$$

$$u(x, 0) = \varphi_1(x), \quad u_t(x, 0) = \varphi_2(x), \quad 0 \leq x \leq l, \quad (3)$$

where $\Omega = \{0 < x < l, 0 < t \leq T\}$; $a, l, T > 0$ are the given numbers; $\delta(\cdot)$ is the Dirac function; $\varphi_i(x) \in L_2(0, l), i = 1, 2$, are the given functions; $s_k(t) \in [0, l]$; $p(t) = (p_1(t), p_2(t), \dots, p_n(t)) \in L_2(0, T; R^n)$, $s(t) = (s_1(t), s_2(t), \dots, s_n(t)) \in L_2(0, T; R^n)$ are the control functions.

The pair of functions $\mathcal{G} = (p(t), s(t))$ is called the control. The Hilbertian space of the pairs $\mathcal{G} = (p(t), s(t))$ with the scalar product

$$(\mathcal{G}^1, \mathcal{G}^2)_{\mathcal{H}} = \int_0^T [p^1(t)p^2(t) + s^1(t)s^2(t)] dt$$

and the norm $\|\mathcal{G}\|_{\mathcal{H}} = \sqrt{(\mathcal{G}, \mathcal{G})_{\mathcal{H}}} = \sqrt{\|p\|_{L_2}^2 + \|s\|_{L_2}^2}$, where $\mathcal{G}^k = (p^k, s^k), k = 1, 2$, is denoted for brevity by $H = L_2(0, T; R^n) \times L_2(0, T; R^n)$.

In the space H , we introduce a set of permissible controls

$$V = \{(p, s) \in H : 0 \leq p_i(t) \leq A_i, \quad 0 \leq s_i(t) \leq B_i \leq l, i = \overline{1, n}\}, \quad (4)$$

and consider the functional

$$\begin{aligned}
J(\mathcal{G}) = & \alpha_1 \int_0^l [u(x, T) - y_1(x)]^2 dx + \\
& + \alpha_2 \int_0^l [u_t(x, T) - y_2(x)]^2 dx + \\
& + \sum_{k=1}^n \left\{ \beta_1 \int_0^T [p_k(t) - \tilde{p}_k(t)]^2 dt + \right. \\
& \left. + \beta_2 \int_0^T [s_k(t) - \tilde{s}_k(t)]^2 dt \right\}, \quad (5)
\end{aligned}$$

where $\alpha_1, \alpha_2 \geq 0, \alpha_1 + \alpha_2 > 0, \beta_1, \beta_2 > 0$ are the given parameters; $A_i > 0, B_i > 0, i = \overline{1, n}$, are the given numbers; $y_i(x) \in L_2(0, l), i = \overline{1, 2}$, $\omega = (\tilde{p}(t), \tilde{s}(t)) \in H$, $\tilde{p}(t) = (\tilde{p}_1(t), \tilde{p}_2(t), \dots, \tilde{p}_n(t))$, $\tilde{s}(t) = (\tilde{s}_1(t), \tilde{s}_2(t), \dots, \tilde{s}_n(t))$ are the given functions.

It is required to find a permissible control $\mathcal{G} = (p(t), s(t))$ from the set V and the appropriate solution $u(x, t)$ mixed problem (1)-(3) such that functional (5) accepts the last possible value.

Below everywhere the conditions accepted in this item are considered as the fulfilled.

3. Correctness of problem statement

Determination of the function

$u(x, t) = u(x, t; \mathcal{G})$ from conditions (1)-(3) under the given control $\mathcal{G} \in V$ is called the reduced problem.

Definition 3.1. Under the generalized solution of the reduced problem (1)-(3) corresponding to the control $\mathcal{G} = (p(t), s(t)) \in V$ we understand the function $u = u(x, t) \in H^1(\Omega)$, satisfying the integral identity

$$\begin{aligned}
& \int_0^l \int_0^T [-u_t \eta_t + a^2 u_x \eta_x] dx dt = \int_0^l \varphi_2(x) \eta(x, 0) dx + \\
& + \sum_{k=1}^n \int_0^T p_k(t) \eta(s_k(t), t) dt, \quad (6)
\end{aligned}$$

$$\forall \eta = \eta(x, t) \in H^1(\Omega) \text{ and } \eta(x, T) = 0.$$

It follows from the results established in [10-11] that for each fixed $\mathcal{G} \in V$ the reduced problem (1)-(3) has a unique solution from $H^1(\Omega)$. Let conditions accepted in the formulation of the problem be satisfied. Then, problem (1)-(5) has at least one solution. It should be noted that for $\beta_j = 0, j = \overline{1, 2}$, problem (1)-(5) is incorrect in the classical sense [12]. However, the following theorem is valid.

Theorem 3.1. There exists a dense subset K of the space H such that problem (1)-(5) has a unique solution for any $\omega \in K$ under $\alpha_i > 0, \beta_i > 0, i = \overline{1, 2}$.

At the proof of the theorem it is used the continuity of the functional

$$\begin{aligned}
J_0(\mathcal{G}) = & \alpha_1 \int_0^l [u(x, T) - y_1(x)]^2 dx + \\
& + \alpha_2 \int_0^l [u_t(x, T) - y_2(x)]^2 dx.
\end{aligned}$$

Further from Bido's [13] theorem, the statement of the theorem follows.

4. Necessary optimality conditions

Let's $\psi = \psi(x, t) \in H^2(\Omega)$ be the generalized solution of the problem conjugate to (1)-(3):

$$\psi_{tt} = a^2 \psi_{xx}, \quad (x, t) \in \Omega, \quad (7)$$

$$\psi_x(0, t) = 0, \quad \psi_x(l, t) = 0, \quad 0 \leq t < T, \quad (8)$$

$$\begin{aligned}
\psi(x, T) &= 2\alpha_2 [u_t(x, T) - y_2(x)], \quad 0 \leq x \leq l, \\
\psi_t(x, T) &= -2\alpha_1 [u(x, T) - y_1(x)], \quad 0 \leq x \leq l. \quad (9)
\end{aligned}$$

Definition 4.1. Under the generalized solution of the problem (7)-(9) corresponding to the control $\mathcal{G} = (p(t), s(t)) \in V$ we understand the function $\psi = \psi(x, t) \in H^2(\Omega)$, satisfying the integral identity

$$\begin{aligned} & \int_0^l \int_0^T (\Phi_{tt} - a^2 \Phi_{xx}) \Psi dx dt = \\ & = \int_0^l 2\alpha_1 (u(x, T) - y_1(x)) \Phi(x, T) dx + \\ & + \int_0^l 2\alpha_2 (u_t(x, T) - y_2(x)) \Phi_t(x, T) dx, \quad (10) \end{aligned}$$

$\forall \Phi = \Phi(x, t) \in H^2(\Omega)$ and $\Phi(x, 0) = \Phi_t(x, 0) = 0$.

The conjugate problem (7)-(9) is a mixed problem for the linear parabolic equation. If in (7)-(9) we take instead of t the new independent variable $\tau = T - t$, then we get a boundary problem of the same type as (1)-(3). It follows, therefore, from the facts established for problem (1)-(3) that for each $\mathcal{G} = (p(t), s(t)) \in V$ problem (1)-(3) has a unique solution from $H^2(\Omega)$.

The function

$$\begin{aligned} H(t, \psi, \mathcal{G}) = & - \sum_{k=1}^n [\psi(s_k(t), t) p_k(t) + \\ & + \beta_1 (p_k(t) - \tilde{p}_k(t))^2 + \beta_2 (s_k(t) - \tilde{s}_k(t))^2] \end{aligned} \quad (11)$$

is called the Hamilton-Pontryagin function of problem (1)-(5). The following theorem is valid.

Theorem 4.1. *If $\psi(x, t) \in H^2(\Omega)$ is the solution of the boundary value problem (7)-(9), functional (5) is Frechet-differentiable on the set V , and the following relation is valid for its gradient:*

$$J'(\mathcal{G}) = \left(\frac{\partial J(\mathcal{G})}{\partial p}, \frac{\partial J(\mathcal{G})}{\partial s} \right) = - \left(\frac{\partial H}{\partial p}, \frac{\partial H}{\partial s} \right), \quad (12)$$

where

$$\begin{aligned} \frac{\partial H}{\partial p} &= \left(\frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2}, \dots, \frac{\partial H}{\partial p_n} \right), \quad \frac{\partial H}{\partial s} = \left(\frac{\partial H}{\partial s_1}, \frac{\partial H}{\partial s_2}, \dots, \frac{\partial H}{\partial s_n} \right), \\ \frac{\partial H}{\partial p_k} &= -\psi(s_k(t), t) - 2\beta_1 (p_k(t) - \tilde{p}_k(t)), \\ \frac{\partial H}{\partial s_k} &= -\psi_x(s_k(t), t) p(t) - 2\beta_2 (s_k(t) - \tilde{s}_k(t)), \quad k = \overline{1, n}. \end{aligned}$$

This theorem is proved by transformation of the increment of functional and estimation of the residual term of this increment.

For the conjugated problem takes place

Theorem 4.2. *Let the functions $u^*(x, t)$ and $\psi^*(x, t)$, respectively, are solutions of problems (1)-(3) and (7)-(9) for $\mathcal{G} = \mathcal{G}^* = (p^*(t), s^*(t)) \in V$. Then for the control $\mathcal{G}^* \in V$ to be optimal, the condition*

$$H(t, \psi^*, \mathcal{G}^*) = \max_{\mathcal{G} \in V} H(t, \psi^*, \mathcal{G}), \quad \forall (x, t) \in \Omega, \quad (13)$$

must be satisfied.

In virtue of the well-known theorem [14, p. 28] for optimality of the control $\mathcal{G}^* = (p^*(t), s^*(t)) \in V$, it is necessary to satisfy the inequality

$$\langle J'(\mathcal{G}^*), \mathcal{G} - \mathcal{G}^* \rangle_H \geq 0, \quad \forall \mathcal{G} \in V. \quad (14)$$

Using expression (12) of the gradient of the functional, we calculate the gradient of the functional (5) and substitute it in (14), and receive

$$\begin{aligned} \langle J'(\mathcal{G}^*), \mathcal{G} - \mathcal{G}^* \rangle_H = & \sum_{k=1}^n \int_0^T \left\{ \left[\psi^*(s_k^*(t), t) + 2\beta_1 (p_k^*(t) - \tilde{p}_k(t)) \right] \times \right. \\ & \times (p_k(t) - p_k^*(t)) + \\ & + \left[\psi_x^*(s_k^*(t), t) p_k^*(t) + 2\beta_2 (s_k^*(t) - \tilde{s}_k(t)) \right] \times \\ & \times (s_k(t) - s_k^*(t)) \left. \right\} dt \geq 0, \quad \forall \mathcal{G} = (p(t), s(t)) \in V, \quad (15) \end{aligned}$$

where $\psi^*(x, t)$ is the solution of problem (7)-(9) for $\mathcal{G} = \mathcal{G}^* = (p^*(t), s^*(t)) \in V$.

As a result the following theorem is proved

Theorem 4.3. *Let all conditions of Theorem 4.2 be satisfied. Then for the control $\mathcal{G}^* = (p^*(t), s^*(t)) \in V$ to be optimal, the condition (15) for all $\mathcal{G} = (p(t), s(t)) \in V$ must be satisfied.*

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DALĞA TƏNLİYİ ÜÇÜN MƏNBƏLƏRİN HƏRƏKƏTİNİN İDARƏ EDİLMƏSİ

R.A. Teymurov

Məqalədə vəziyyəti dalğa tənliyi ilə təsvir olunan proseslərdə hərəkət edən mənbələrin optimal idarə edilməsi məsələsinin variasiya üsulu ilə həllinə baxılmışdır. Burada optimal idarəetmə məsələsinin həllinin varlığı və yeganəliyi isbat edilmiş, optimallıq üçün nöqtəvi və maksimum prinsipi şəklində zəruri şərt alınmış, məqsəd funksionalının Freşe mənada diferensiallanan olması üçün kafi şərt və onun qradiyenti üçün ifadə tapılmışdır.

Açar sözlər: hərəkət edən mənbələr, inteqral eyniliyi, Hamilton–Pontryagin funksiyası, optimallıq üçün zəruri şərtlər, maksimum prinsipi

УПРАВЛЕНИЕ ДВИЖЕНИЕМ ИСТОЧНИКОВ ДЛЯ ВОЛНОВОГО УРАВНЕНИЯ

Р.А.Теймуров

В работе исследуется задача оптимального управления движением источников для колебательной системы, состояние которых описываются волновым уравнением. Для рассмотренной ниже задачи оптимального управления доказана теорема существования и единственности решения, найдены достаточные условия дифференцируемости по Фреше критерия качества и получено выражение для его градиента, получены необходимые условия оптимальности в виде точечного и интегрального принципов максимума.

Ключевые слова: подвижные источники, интегральное тождество, функция Гамильтона–Понтрягина, необходимые условия оптимальности, принцип максимума