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Mathematics

GAUSSIAN SEMI-MARKOVIAN RANDOM WALK WITH A SMALL DRIFT AND WITH DELAY ON LEVEL ZERO

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In this paper the Gaussian semi-Markovian random walk process with discrete interference of chance, assuming that the discrete interference of chance described by sequence of independent random variables with gamma distribution with parameters (α, λ) , $\alpha > 1, \lambda > 0$ are investigated. Asymptotic formulas with the approximate coefficients of the first four moments of the process are obtained.

Keywords: Gaussian random walk, ergodic distribution, asymptotic expansion, ladder variables

Introduction

Let $\{\xi_n\}$, $\{\eta_n\}$ and $\{\theta_n\}$, $n \geq 1$ – be three independent sequences of random variables defined on some probability space $(\Omega, \mathfrak{F}, P)$, such that variables in each sequence are independent and identically distributed. Assume that random variables ξ_n and θ_n take only positive values and η_n has normal distribution with parameters $(b, 1)$.

Also introduce a sequence of independent random variables $\{\zeta_n\}$, $n \geq 1$, which describes the discrete interference of chance. Assume that the random variables ζ_n have a gamma distribution with parameters (α, λ) , $\alpha > 1, \lambda > 0$.

Define renewal process $\{T_n\}$ and random walk $\{S_n\}$:

$$T_n = \sum_{i=1}^n \xi_i, \quad S_n = \sum_{i=1}^n \eta_i, \quad T_0 = S_0 = 0, \quad n = 1, 2, \dots$$

Introduce also a sequence of integer random variables N_n :

$$N_0 = 0; \quad N_1 = N(z) = \inf \{k \geq 1: z - S_k < 0\},$$

$$N_n \equiv N_n(\zeta_n) = \inf \{k \geq N_{n-1} + 1: \zeta_n - (S_k - S_{N_n}) < 0\},$$

$$n = 1, 2, \dots$$

Assume that $\inf\{\emptyset\} = +\infty$.

Introduce also following random variables:

$$\tau_0 = 0, \quad \tau_1 = T_{N_1}, \quad \gamma_1 = \tau_1 + \theta_1,$$

$$\tau_2 = \gamma_1 + T_{N_2} - T_{N_1}, \quad \gamma_2 = \tau_2 + \theta_2,$$

$$\dots$$

$$\tau_n = \gamma_{n-1} + T_{N_n} - T_{N_{n-1}}, \quad \gamma_n = \tau_n + \theta_n, \quad n = 1, 2, \dots$$

Define so called counting process

$$\nu(t) = \max \{n \geq 0: T_n \leq t\},$$

which means the number of jumps of the process $X(t)$ on the interval $[0, t]$.

Now write an analytical form of the random process $X(t)$:

$$X(t) = \max \{0, \zeta_n - (S_{\nu(t)} - S_{N_n})\},$$

$$\text{at } \gamma_n \leq t < \gamma_{n+1}, \quad n = 0, 1, 2, \dots$$

Our purpose is to obtain an asymptotic expansion for the four moments of the ergodic distribution of process $X(t)$. Note, this problem for a particular case, is consider in [3].

2. Main results

As seen from the analytical form of the process $X(t)$, one of its components is a random walk

$$S_0 = 0, \quad S_n = \sum_{i=1}^n \eta_i, \quad n \geq 1,$$

where $\{\eta_n\}$, $n \geq 1$ is a sequences of independent

and identically distributed random variables.

Assume $\nu_1^+ = \min\{k \geq 1: S_k > 0\}$, $\chi_1^+ = S_{\nu_1^+}$.

Note that the random variables ν_1^+ and χ_1^+ are called, respectively, the first ladder epoch moment and the first ladder height of the random walk $\{S_n\}$, $n \geq 1$, (see [4, 391]).

Let's $\{\nu_n^+\}$ and $\{\chi_n^+\}$, $n \geq 1$, independent sequence of positive random variables, and the variables of each in a sequence of independent and identically distributed as ν_1^+ and χ_1^+ , accordingly. By definition

$$\nu_{m+1}^+ = \min\left\{k \geq 1: S_{\nu_m^+ + k} > \chi_m^+\right\};$$

$$\chi_{m+1}^+ = S_{\nu_{m+1}^+} = \sum_{i=1}^{\nu_{m+1}^+} \eta_i; m=1,2,\dots$$

Applying the method to use [1] obtained the following asymptotic expansions for the first four moments of the ergodic distribution process $X(t)$ at $\lambda \rightarrow 0$:

Theorem 2.1. Let's $\{\xi_n\}$, $\{\eta_n\}$, $\{\zeta_n\}$ and $\{\theta_n\}$, $n \geq 1$ are independent sequence of random variables and within each sequence of random variables are independent and identical distributed. Let satisfied the following conditions:

1) $E\xi_1 < \infty$; 2) $E\theta_1 < \infty$; 3) $E|\eta_1|^5 < \infty$; 4) random variables ζ_n has gamma distribution with parameters (α, λ) , $\alpha > 1, \lambda > 0$. Then the following asymptotic expansions for the first four moments of the ergodic distribution process $X(t)$, at $\lambda \rightarrow 0$ is true:

$$E(X^k) = \frac{\alpha_{[k]}}{(k+1)\lambda^k} + \left[\frac{\alpha_{[k-1]}}{2} m_{21} - \frac{\alpha_{[k]}}{(2k+2)\alpha} (\mu_{21} + K m_1) \right] \frac{1}{\lambda^{k-1}} + o\left(\frac{1}{\lambda^{k-1}}\right); \quad (2.1)$$

where $\alpha_{[k]} = (\alpha+1)(\alpha+2)\cdots(\alpha+k)$, $k = \overline{1,4}$;

$$m_{k1} = \frac{m_k}{m_1}; \mu_{k1} = \frac{\mu_k}{\mu_1}; \mu_k = E(\chi_1^+)^k,$$

$K = E\theta_1/E\xi_1$ - delay coefficient.

From the asymptotic expansions (2.1) obtained for ergodic moments of the process $X(t)$ can be seen that the coefficients of these expansions contain parameters $\mu_k = E(\chi_1^+)^k$, $k = 1, 2$, which are the first moments of the ladder height of a random walk $\{S_n\}$, $n \geq 0$. In the general case it is impossible to determine the parameters $\mu_k = E(\chi_1^+)^k$, $k = 1, 2$, even if you set the distribution of the components $\{\eta_n\}$, $n \geq 1$ of a random walk $\{S_n\}$, $n \geq 0$.

In what follows we will designate $\mu_k(0)$ and $\mu_k(b)$, accordingly, the moment of k order of the first ladder height when the mathematical expectation of a random walk components $\{\eta_n\}$, $n \geq 1$ is equal to zero and b .

Our purpose is to obtain the asymptotic expansion for moments of ladder heights Gaussian random walk with a small drift, ie, when $b \rightarrow 0$.

For this purpose, we will use the next lemma from [3].

Lemma 2.1. For the characteristics $\mu_{21}(b)$ of a Gaussian random walk following holds expansion at $b \rightarrow 0$:

$$\mu_{21}(b) \equiv \frac{\mu_2(b)}{\mu_1(b)} = \mu_{21}(0) + \frac{1}{6} (4\mu_{31}(0) - 3\mu_{21}^2(0))b + o(b), \quad (2.7)$$

where $\mu_1(0) = E(\chi_1^+) = \frac{\sqrt{2}}{2}$, $\mu_2(0) = E(\chi_1^{+2}) = \frac{L}{\sqrt{\pi}}$,

$$\mu_3(0) = E(\chi_1^{+3}) = \frac{3\sqrt{2}}{8\pi} (\pi + 2L^2),$$

$$L = \lim_{n \rightarrow \infty} \left[2\sqrt{n} - \sum_{m=1}^n \frac{1}{m} \right] \text{ (see [2]).}$$

Now, using Lemma 2.1, we can write the approximate formulas for the coefficients of the asymptotic expansion (2.1) for moments of ergodic distribution of process $X(t)$ in the case

$$S_n = \sum_{i=1}^n \eta_i, n \geq 1 \text{ where a Gaussian random}$$

walks with a small drift, ie, when $b \equiv E\eta_1 \rightarrow 0$:

Lemma 2.2. Let the conditions of Theorem 2.1, and η_1 has a normal distribution with the parameters $(b, 1)$, $b > 0$. Then we have the following approximate formula for the coefficients of the asymptotic expansions of Theorem 2.1 at $b \rightarrow 0$:

$$A_k(\alpha, b) = \frac{1}{2b} \alpha_{[k-1]} - \mu_{21}(0) \frac{\alpha_{[k]}}{(2k+2)\alpha} - \left(\frac{\alpha_{[k-1]}}{2} + \frac{1}{6} (4\mu_{31}(0) - 3\mu_{21}^2(0) + K) \frac{\alpha_{[k]}}{(2k+2)\alpha} \right) b + o(b), k = \overline{1, 4}.$$

Proof. In case of normal distribution with parameters $(b, 1)$ $m_1 = b$ and $m_{21} = \frac{b}{2} + \frac{1}{2b}$. Taking into account as well (2.7) for the coefficients of the asymptotic expansion (2.1) we have:

$$\begin{aligned} A_k(\alpha, b) &= \frac{\alpha_{[k-1]}}{2} m_{21} - \frac{\alpha_{[k]}}{(2k+2)\alpha} (\mu_{21}(b) + K m_1) = \\ &= \left(\frac{b}{2} + \frac{1}{2b} \right) \alpha_{[k-1]} - \left(\mu_{21}(0) + \frac{1}{6} (4\mu_{31}(0) - \right. \\ &\quad \left. - 3\mu_{21}^2(0) + K) b + o(b) \right) \frac{\alpha_{[k]}}{(2k+2)\alpha} = \\ &= \frac{1}{2b} \alpha_{[k-1]} - \mu_{21}(0) \frac{\alpha_{[k]}}{(2k+2)\alpha} - \left(\frac{\alpha_{[k-1]}}{2} + \right. \end{aligned}$$

$$\left. + \frac{1}{6} (4\mu_{31}(0) - 3\mu_{21}^2(0) + K) \frac{\alpha_{[k]}}{(2k+2)\alpha} \right) b + o(b).$$

Lemma 2.2 is proved. $\square$

The following theorem holds:

Theorem 2.2. Let the conditions of Theorem 2.1. Also, assume that η_n have normal distribution with parameters $(b, 1)$.

Then, when $\lambda \rightarrow 0$ we have the following asymptotic expansions with approximate coefficients for the first and second moments of the ergodic distribution of Gaussian semi-Markov walks with a small drift:

$$E(X^k) = \frac{\alpha_{[k]}}{(k+1)\lambda^k} + A_k(\alpha, b) \frac{1}{\lambda^{k-1}} + o\left(\frac{1}{\lambda^{k-1}}\right),$$

where $A_k(\alpha, b)$, $k = \overline{1, 4}$ defined from Lemma 2.2.

Proof. The statement of the theorem follows immediately from Theorem 2.1 with Lemma 2.2.

This Theorem 2.2 is proved.

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**KİÇİK MEYİLLİ VƏ SIFIR SƏVİYYƏSİNDƏ GECİKMƏYƏ MALİK QAUSS
SEMİ-MARKOV DOLAŞMA PROSESİ****R.T. Əliyev, H.Ə.Cəfərova**

Təqdim olunan məqalədə (α, λ) , $\alpha > 1, \lambda > 0$ parametrlı qamma paylanmaya malik diskret müdaxiləli kiçik meyilli Qauss semi-Markov dolaşma prosesi tədqiq olunmuşdur. Prosesin ilk dörd erqodik momentləri üçün təqribi əmsallı asimptotik ayrılışlar əldə edilmişdir.

Açar sözlər: Qauss təsadüfi dolaşması, erqodik paylanma, asimptotik ayrılış, pilləvari kəmiyyətlər

**ГАУССОВСКОЕ ПОЛУМАРКОВСКОЕ БЛУЖДЕНИЕ С МАЛЫМ
СНОСОМ И С ЗАДЕРЖКОЙ НА УРОВНЕ НУЛЬ****Р.Т. Алиев, Х.А.Джафарова**

В данной работе исследуется Гауссовское полумарковское блуждание с дискретным вмешательством случая, в предположении, что дискретное вмешательство описывается последовательностью независимых случайных величин с гамма-распределением с параметрам, (α, λ) , $\alpha > 1, \lambda > 0$. Получены асимптотические формулы с приближенными коэффициентами для первых четырех моментов процесса.

Ключевые слова: гауссовское случайное блуждание, эргодическое распределение, асимптотическое разложение, лестничные величины