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Mathematics

ON NECESSARY OPTIMALITY CONDITIONS IN DISCRETE SYSTEMS

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In the paper, a rather necessary optimality condition is obtained and is shown that in some special cases, this condition contains known and also a new necessary optimality condition.

Keywords: discrete systems, discrete principle of maximum, permissible process

Introduction

A.G. Butkovskii [1] first showed that unlike the continuous case, direct transference of L.S. Pontryagin's maximum principle to discrete systems in general case is impossible. Naturally, such characteristics of these systems is of theoretical interest for researchers. In this connection, a number of researchers [2,3,4 and others], by imposing additional conditions on the data of the problem under consideration (for example, condition of convexity of the set of permissible velocities of the system, convexity in direction, etc.) established that subject to these conditions the maximum principle is valid for discrete optimal systems. After appearance of the paper[1], many researchers tried to prove the maximum principle in the weakened form (local maximum, stationarity) and to obtain higher order optimality condition. In spite of the fact that many important results have been obtained in the problems of optimization of discrete systems [2-14 and so on] the issue of necessary optimality conditions is far from being completed. This, in first turn, concerns the generality of initial data of the problem under consideration. Therefore, there is a necessity for thorough study of optimization of discrete control problems.

In the present paper, we consider a discrete optimization problem with rather general initial data (without assumptions as convexity and smoothness), and taking into account the characteristics of the discrete system, obtain a necessary optimality condition that is not formulated by the Hamilton-Pontryagin function. It should be noted that this optimality condition is characterized by three peculiarities: 1) it al-

lows to replace the many-dimensional optimization problem by a sequence of problems of less dimension, i.e. the problem under consideration for each discrete time is reduced to minimization of the function of r variables, where r is the size of the control vector; 2) by considering some particular optimization problems (including the problems with quadratic and nonsmooth test functionals) it is proved that this condition contains the known and also new effective optimality conditions; 3) this condition has a wide scope, is stronger than the discrete maximum principle, linearized principle of maximum, Euler-type optimality condition and local optimality conditions formulated by different types of star vicinities and therefore it is very useful when applying the contraction of the controls set suspicious on optimality that are distinguished by more constructive local conditions.

Problem statement

Assume that it is required to minimize the functional

$$S(u) = \Phi(x(t_1)) \quad (1)$$

on the solutions of the discrete system

$$x(t+1) = f(x(t), u(t), t), t \in T, x(t_0) = x^*, \quad (2)$$

at the constraint

$$u(t) \in U(t), t \in T. \quad (3)$$

Here $x = (x_1, \dots, x_n)'$ is the state vector ('(prime)' is the transposition operation), $u = (u_1, \dots, u_r)'$ is a control function, t is a time

(discrete), x^* is the given vector, $T = \{t_0, t_0 + 1, \dots, t_1 - 1\}$; $U(t)$, $t \in T$ is the set of r - dimensional Euclidean space E^r ; $\Phi(x)$, $x \in E^n$, $f(x, u, t)$, $(x, u, t) \in E^n \times E^r \times [t_0, t_1]$ are continuous functions. We call the control $u(t)$, $t \in T$ responding to constraint (3), permissible; the permissible control $u(t)$, $t \in T$ and the solution $x(t)$, $t \in T \cup \{t_1\}$ of system (2), corresponding to it, for which functional (1) accepts the least value, call optimal. Herewith each pair $(u(t), x(t))$ is called an optimal process.

Optimality conditions

Let $x^0(t)$, $t \in T \cup \{t_1\}$ be the solution of system (2), corresponding to the permissible control $u^0(t)$, $t \in T$. Following [15], we introduce the set:

$$U[x^0(\cdot)](t) = \{\hat{u} \in U(t) : f(x^0(t), \hat{u}, t) = f(x^0(t), u^0(t), t)\}, t \in T. \quad (4)$$

Theorem. Let $u^0(t)$, $t \in T$ be a permissible control, $x^0(t)$, $t \in T$ be the solution of system (2) corresponding to it. Then:

a) for the optimality of the control $u^0(t)$, $t \in T$, it is necessary that for each $\hat{u}(t) \in U[x^0(\cdot)](t)$, $t \in T$ for all

$\alpha = (\theta, \nu) \in T \setminus \{t_1 - 1\} \times U(\theta)$,
 $\alpha_k = (\theta_k, \tilde{\nu}) \in \{\theta_1, \theta_2, \dots\} \cap T \times U(\theta_k)$ the following inequality to be fulfilled:

$$\Phi(x^0(t_1) + z(t_1; \hat{u}, x^0, \alpha, \alpha_k)) - \Phi(x^0(t_1)) \geq 0, \quad (5)$$

b) if: 1) in addition to assumptions of point 1, the set T consists of at most two points ($T = \{t_0, t_0 + 1\}$, $t_1 = t_0 + 2$) and 2) along even if one control $\hat{u}(t) \in U[x^0(\cdot)](t)$, $t \in T$, for example along the control $u^0(t)$, $t \in T$ inequality (5) is fulfilled for all $\alpha = (t_0, \nu)$, $\nu \in U(t_0)$, $\alpha_1 = (t_0 + 1, \tilde{\nu})$, $\tilde{\nu} \in U(t_0 + 1)$, then the control $u^0(t)$, $t \in T$ is optimal.

Here the sets $U[x^0(\cdot)](t)$, $t \in T$ are determined according to (4), and $z(t; u, x^0, \alpha, \alpha_k)$ is the solution of the following system corresponding to the process: $(\hat{u}(t), x^0(t))$:

$$\begin{cases} z(t+1; u^0, x^0, \alpha, \alpha_k) = \\ = f(x^0(t) + z(t; u^0, x^0, \alpha, \alpha_k), u^0(t), t) - \\ - f(x^0(t), u^0(t), t), \\ t \in \{\theta_{k+1}, \theta_{k+2}, \dots\} \cap T, \\ z(\theta_{k+1}; u^0, x^0, \alpha, \alpha_k) = \\ = f(x^0(\theta_k) + z(\theta_k; u^0, x^0, \alpha, \tilde{\nu}, \theta_k) - \\ - f(x^0(\theta_k), u^0(\theta_k), \theta_k), \\ \text{если } t = \theta_k \in T, \\ z(t+1; u^0, x^0, \alpha) = \\ = f(x^0(t) + z(t; u^0, x^0, \alpha), u^0(t), t) - \\ - f(x^0(t), u^0(t), t), \\ t \in \{\theta_1, \theta_2, \dots, \theta_{k-1}\} \cap T, \\ z(\theta_1; u^0, x^0, \alpha) = \Delta_\nu f(x^0(\theta), u^0(\theta), \theta), \theta \in T, \end{cases} \quad (6)$$

where $\alpha = (\theta, \nu) \in T \times U(\theta)$, $\alpha_k = (\theta_k, \tilde{\nu}) \in \{\theta_1, \theta_2, \dots\} \cap T \times U(\theta_k)$, $\theta_k = \theta + k$, $k = 1, 2, \dots$

It should be noted that this theorem, as a special case, contains a number of known and also new effective optimality conditions. Cite some of them.

Corollary 1. Let in problem (1)-(3): 1) the vector-function $f(x, u, t)$ be linear with respect to x , i.e. $f(x, u, t) = A(u, t)x + b(u, t)$, where $A(u, t)$ is a matrix function of order $n \times n$, $b(u, t)$ is n -dimensional vector-function and 2) the test functional has the form:

$\Phi = c'x + x'Dx$, $x \in E^n$, where c is n -dimensional vector, a D is a matrix of order $n \times n$. Then for the optimality of the process $u^0(t)$, $t \in T$ it is necessary that for each $\hat{u}(t) \in U[x^0(\cdot)](t)$, $t \in T$ and for all $(\theta, \nu) \in T \times U(\theta)$ the following inequality to be fulfilled

$$\Delta_{\nu} H(\hat{\psi}(\theta), x^0(\theta), u^0(\theta), \theta) + \Delta_{\nu} f'(x^0(\theta), u^0(\theta), \theta) \hat{\Psi}(\theta) \Delta_{\nu} f(x^0(\theta), u^0(\theta), \theta) \leq 0, \quad (7)$$

where

$$U[x^0(\cdot)](t) = \{\hat{u} \in U(t) : A(\hat{u}, t)x^0(t) + b(\hat{u}, t) = A(u^0(t), t)x^0(t) + b(u^0(t), t)\}, t \in T; \quad (8)$$

$$\begin{aligned} d^0 &= c + (D + D')x^0(t_1), \Delta_{\nu} f(x^0(\theta), u^0(\theta), \theta) = \\ &+ \Delta_{\nu} A(u^0(\theta), \theta)x^0(\theta) + \Delta_{\nu} b(u^0(\theta), \theta), \\ H(\hat{\psi}, x, u, t) &= \hat{\psi}' f(x, u, t) = \\ &= \hat{\psi}' [A(u, t)x + b(u, t)] \end{aligned} \quad (9)$$

$$\hat{\psi}(t-1) = A'(\hat{u}(t), t)\hat{\psi}(t), t \in T, \hat{\psi}(t_1-1) = -d^0, \quad (10)$$

$$\begin{aligned} \hat{\Psi}(t-1) &= A'(\hat{u}(t), t)\hat{\Psi}(t)A(\hat{u}(t), t), t \in T, \\ \hat{\Psi}(t_1-1) &= -\frac{1}{2}(D + D') \end{aligned} \quad (11)$$

Corollary 2. Let in problem (1)-(3) $f(x, u, t) = A(u, t)x + b(u, t)$, $\Phi(x) = c'x + |d'x|$, where $A(u, t)$ is a matrix function of order $n \times n$, $b(u, t)$ is n -dimensional vector-function, c and d are n -dimensional vectors. Then for the optimality of the admissible process $(u^0(t), x^0(t))$ it is necessary that for each $\hat{u}(t) \in U[x^0(\cdot)](t)$, $t \in T$ and for all $\theta \in T$, $\nu \in U(\theta)$ the following inequality to be fulfilled

$$\begin{aligned} |d'x^0(t_1)| + \Delta_{\nu} H(\hat{\psi}_c(\theta), x^0(\theta), u^0(\theta), \theta) \leq \\ \leq |d'x^0(t_1)| - \Delta_{\nu} H(\hat{\psi}_d(\theta), x^0(\theta), u^0(\theta), \theta), \end{aligned} \quad (12)$$

where the sets $U[x^0(t)](t)$, $t \in T$ are determined according to (8), while $H(\hat{\psi}, x, u, t)$ according to (9), $\hat{\psi}_c(t)$, $\hat{\psi}_d(t)$ are the solutions of system (10) with appropriate initial conditions $\hat{\psi}_c(t_1-1) = -c$, $\hat{\psi}_d(t_1-1) = -d$.

Conclusion

It should be noted that the necessary optimality condition (5) is rather general and has a wider scope than the earlier known necessary optimality conditions [1,4,6-9, etc.]. We also note that the optimality condition of type (7), for $\hat{u}(t) = u^0(t)$, $t \in T$ was obtained by another way in [8], while the optimality condition (12), as far as we know, has no analogue.

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DİSKRET SİSTEMLƏRDƏ OPTİMALLIĞIN ZƏRURİ ŞƏRTLƏRİ HAQQINDA **M.C. Mərdanov, S.T. Məlik**

Məqalədə diskret sistemlərdə təsvir olunan idarəetmə məsələləri üçün optimallığın daha ümumi zəruri şərti alınmışdır. Həmçinin bu zəruri şərtədən məlum optimallıq şərtinin və yeni optimallıq şərtinin alınması göstərilmişdir.

Açar sözlər: diskret sistemlər, diskret maksimum prinsipi, mümkün proses

О НЕОБХОДИМЫХ УСЛОВИЯХ ОПТИМАЛЬНОСТИ В ДИСКРЕТНЫХ СИСТЕМАХ **М. Дж. Марданов, С. Т. Малик**

В работе получено довольно общее необходимое условие оптимальности и показано, что это условие, в некоторых частных случаях, содержит известные, а также новое необходимое условие оптимальности.

Ключевые слова: дискретные системы, дискретный принцип максимума, допустимый процесс