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ASYMPTOTICAL METHOD FOR DETERMINING THE FRACTIONAL ORDER IN OSCILLATORY SYSTEMS BY MOVING MASS COORDINATES

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Note that in an oscillatory system, when the mass of oil lifted by the plunger is sufficiently large, an asymptotical method is introduced to solve the given problem, taking the inverse of the mass as a small parameter. At this time, the motion of the object is reduced to the Volterra integral equation of the second kind with respect to the displacement of mass coordinates. Based on statistical data, a quadratic functional is constructed. Using the least squares method for determining the fractional order an effective algorithm is proposed and the table with certain fractional orders is presented in the example. The result of the computational algorithm shows that the first variation of the quadratic functional with respect to the fractional order is equal to zero with an accuracy of 10^{-16} . This fractional order is the most effective order. The obtained fractional order satisfies the first variation of the quadratic functional with an accuracy of 10^{-16} .

Keywords: oscillatory system, fractional order differential equation, Volterra integral equation, the least squares method

1. Introduction

It is known that the motion of classical oscillatory systems is described in the simplest case by second-order linear differential equations with constant coefficients. It is known that in recent years, the most widespread method of operation of oil wells [5-7, 9] is the method of operation with rod pumping units. According to recent studies, since the plunger moves inside a Newtonian fluid [1], the motion of the oscillatory system is described by fractional-order second-order differential equations with constant coefficients [2-4].

In the current paper, the oscillatory system with liquid dampers is considered, where the mass of oil lifted by the plunger is sufficiently large. In this case an asymptotical method is introduced to solve this problem, taking the inverse of the mass as a small parameter.

Note that this issue has been discussed before [8]. In that work the corresponding Volterra integral equation of the second kind is discretized. Taking into account the solution of this equation and the statistical data in the quadratic functional, this functional is minimized by dividing the segment in half on the interval where the fractional order is included. In the

example, the first variation of the functional cannot be brought closer to zero below the order of 10^{-4} . This shows that the method [8] is not very effective.

2. Problem Statement

Let the motion of an object be described by the following system of linear differential equations with fractional derivative [11-13,15]

$$m\ddot{y}(x) + aD^\alpha y(x) + by(x) = f(x), \quad x \geq x_0 > 0, \quad (1)$$

with initial conditions

$$y(x_0) = 0, \quad \dot{y}(x_0) = y_1, \quad (2)$$

where $\alpha \in (1,2)$, $y(x)$ – the desired function, m, a, b, y_1, x_0 – given parameters, $f(x)$ – external force.

Suppose that the mass m in equation (1) is large enough. Then divide both sides of this equation by m and denote $\frac{1}{m} = \varepsilon$:

$$\ddot{y}(x) + \varepsilon a D^\alpha y(x) + \varepsilon b y(x) = \varepsilon f(x), \quad x \geq x_0 > 0, \quad (3)$$

Let us introduce the following quadratic functional to find α :

$$J(\alpha) = \min_{\alpha} \left(y(l) - \sum_{j=1}^s \frac{y_j}{s} \right)^2, \quad (4)$$

where y_j , $j = \overline{1, s}$ statistical data for finding α , $y(l)$ — the value of the solution to the problem (2)-(3) at the point l .

3. Reduction of the problem to a Volterra integral equation of the second kind

To solve the problem (2)-(4), we integrate the equation (3) twice from x_0 to x and take into account the initial condition (2). Thus, we reduce the problem (2)-(3) to a Volterra integral equation of the second kind with respect to $y(x)$ [14]:

$$y(x) + \int_{x_0}^x K(x-t)y(t)dt = F(x), \quad (5)$$

where

$$K(x-t) = \varepsilon \left[a \frac{(x-t)^{1-\alpha}}{(1-\alpha)!} + b(x-t) \right],$$

$$F(x) = \varepsilon \int_{x_0}^x (x-t)f(t)dt + y_1 x.$$

Now let's introduce the asymptotic separation of the function $y(x)$ as follows:

$$y(x) = \sum_{i=0}^{\infty} \varepsilon^i y^{(i)}(x). \quad (6)$$

If we consider the expression (6) in equation (5), we get:

$$y^{(0)}(x) = y_1 x, \quad (7)$$

$$y^{(1)}(x) = - \int_{x_0}^x \left[a \frac{(x-t)^{1-\alpha}}{(1-\alpha)!} + b(x-t) \right] y_1 t dt + \int_{x_0}^x (x-t)f(t)dt, \quad (8)$$

Let us consider the solutions (7) and (8) in (6):

$$y(x) \approx y_1 x - \varepsilon \int_{x_0}^x \left[a \frac{(x-t)^{1-\alpha}}{(1-\alpha)!} + b(x-t) \right] y_1 t dt + \int_{x_0}^x (x-t)f(t)dt. \quad (9)$$

Further we insert the expression (9) at point l and we consider the obtained expression in functional (4):

$$J(\alpha) = \min_{\alpha} \left(y_1 l - \varepsilon \int_{x_0}^l \left[a \frac{(l-t)^{1-\alpha}}{(1-\alpha)!} + b(l-t) \right] y_1 t dt + \int_{x_0}^l (l-t)f(t)dt \right). \quad (10)$$

To find the parameter α , we find the derivative of functional (10) with respect to α and set it equal to zero:

$$\frac{\partial J(\alpha)}{\partial \alpha} \approx \frac{J(\alpha+h) - J(\alpha)}{h} \approx 0. \quad (11)$$

So, we introduce the following algorithm to solve problems (2)-(4):

Algorithm

- Enter the values of the parameters $\varepsilon, a, b, y_1, f, l, x_0$ included in problem (3)-(4).

• Enter the statistical data $y_p(x)$, $p = \overline{1, s}$.

• Construct the functional (10).

• Using the least squares method, we check condition (11) to determine the parameter α .

4. Numerical Realization

Let's consider the following example [10]:

$$\varepsilon = 10^{-10}, a = 3, b = 1, y_1 = 10^{-10}, f = 8, l = 1, s = 11, \\ y_1(x) = 0, y_2(x) = -0.67, y_3(x) = -0.34, y_4(x) = \\ = 0.81, y_5(x) = 1.22, y_6(x) = 1.44, y_7(x) = 1.57, y_8(x) = \\ = 1.66, y_9(x) = 1.72, y_{10}(x) = 1.77, y_{11}(x) = 1.81.$$

Substituting the above parameters into the problem (2)-(4) and applying these parameters to the above algorithm, we insert the table and we get that the efficient fractional order is $\alpha = 1.1$.

	$\frac{\partial J(\alpha)}{\partial \alpha}, \alpha = 1.1$	$\frac{\partial J(\alpha)}{\partial \alpha}, \alpha = 1.2$	$\frac{\partial J(\alpha)}{\partial \alpha}, \alpha = 1.4$	$\frac{\partial J(\alpha)}{\partial \alpha}, \alpha = 1.85$
$h = 10^{-1}$	$0.9904 \cdot 10^{-17}$	$0.129 \cdot 10^{-16}$	$0.243 \cdot 10^{-16}$	$0.961 \cdot 10^{-14}$
$h = 10^{-2}$	$0.8834 \cdot 10^{-17}$	$0.113 \cdot 10^{-16}$	$0.206 \cdot 10^{-16}$	$0.344 \cdot 10^{-14}$
$h = 10^{-3}$	$0.8738 \cdot 10^{-17}$	$0.1123 \cdot 10^{-16}$	$0.203 \cdot 10^{-16}$	$0.323 \cdot 10^{-14}$
$h = 10^{-4}$	$0.8728 \cdot 10^{-17}$	$0.11224 \cdot 10^{-16}$	$0.20297 \cdot 10^{-16}$	$0.3214 \cdot 10^{-14}$
$h = 10^{-5}$	$0.8727 \cdot 10^{-17}$	$0.11222 \cdot 10^{-16}$	$0.20294 \cdot 10^{-16}$	$0.3213 \cdot 10^{-14}$

5. Conclusion

In this paper, asymptotical method for determining the fractional order in oscillatory systems with liquid dampers is proposed by reducing this equation to the Volterra integral equation of the second kind with respect to the displacement of mass coordinates. The first variation of the effective value of the fractional order obtained by proposed algorithm approaches zero with an accuracy of 10^{-17} . This indicates that the proposed algorithm is more effective.

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HƏRƏKƏT EDƏN KÜTLƏNİN KOORDİNATLARI VASİTƏSİLƏ RƏQSİ SİSTEMLƏRDƏ KƏSR TƏRTİBİN TƏYİNİ ÜÇÜN ASİMPOTOTİK ÜSUL

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Qeyd edək ki, rəqsi sistemlərdə plunjerin qaldırdığı neftin kütləsi kifayət qədər böyük olduqda kütlənin tərsini kiçik parametrlə kimi qəbul edərək, verilmiş məsələnin həlli üçün asimptotik üsul daxil edilir. Bu zaman obyektin hərəkət tənliyi hərəkət edən kütlənin koordinatlarına nəzərən ikinci növ Volterra integral tənliyinə gətirilir. Statistik verilənlərə əsasən kvadrat funksional qurulur. Ən kiçik kvadratlar üsulundan istifadə edərək, kəsr tərtibin təyini üçün effektiv alqoritm daxil edilir və bu alqoritm əsasında cədvəl daxil edilir. Hesablama alqoritmının nəticəsi onu göstərir ki, alınan kəsr tərtib kvadrat funksionalın birinci variasiyasını 10^{-16} dəqiqliklə sıfıra bərabər edən ən effektiv kəsr tərtibdir.

Açar sözlər: rəqsi sistem, kəsr tərtib diferensial tənlik, Volterra integral tənliyi, ən kiçik kvadratlar üsulu

АСИМПТОТИЧЕСКИЙ МЕТОД ОПРЕДЕЛЕНИЯ ДРОБНОГО ПОРЯДКА В КОЛЕБАТЕЛЬНЫХ СИСТЕМАХ С ПОМОЩЬЮ ПЕРЕМЕЩЕНИЯ КООРДИНАТ МАСС

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Отметим, что в колебательной системе, когда масса нефти, поднимаемая плунжером, достаточно велика, вводится асимптотический метод решения поставленной задачи, принимая в качестве малого параметра величину, обратную массе. При этом движение объекта сводится к интегральному уравнению Вольтерра второго рода относительно смещения координат масс. На основе статистических данных строится квадратичный функционал. С использованием метода наименьших квадратов для определения дробного порядка предложен эффективный алгоритм, и в примере представлена таблица с некоторыми дробными порядками. Результат вычислительного алгоритма показывает, что первая вариация квадратичного функционала относительно дробного порядка равна нулю с точностью 10^{-16} . Этот дробный порядок является наиболее эффективным порядком.

Ключевые слова: колебательная система, дифференциальное уравнение дробного порядка, интегральное уравнение Вольтерра, метод наименьших квадратов