

UDC 519.622.2

<https://doi.org/10.59849/0002-3078.2025.1-2.10>

THE METHOD FOR CONSTRUCTING THE ASYMPTOTIC SOLUTION OF THE PERIODIC OPTIMIZATION PROBLEM FOR DISCRETE SYSTEMS WITH LIQUID DAMPERS.

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(Presented by Academician of ANAS F.A.Aliev)

The problem of constructing control and stabilization systems for oil extraction using a rod pump unit is studied with differential equations containing a small parameter in fractional form. An optimal control problem is formulated for them, and in the first approximation, the stabilization problem with small parameters is solved for the given problem.

Keywords: rod pump unit, asymptotics, discretization, Euler-Lagrange equation, algebraic discrete Riccati equation, control, stability, small parameter, fluid dampers, periodic optimization, fractional

Introduction

It is known that the motion equation of an oscillator systems with liquid damper is defined by the following fractional-order linear differential equation [2,4,7]:

$$m_1 y''(x) + aD^\alpha y(x) + by(x) = f(x), \quad 0 \leq x < l, \\ l < x \leq 2l \quad (1)$$

Let $\varepsilon = \frac{1}{m}$ is a small parameter in equation (1) and then

$$y''(x) + \varepsilon aD^\alpha y(x) + \varepsilon by(x) = \varepsilon f(x), \quad 0 \leq x < l, \\ l < x \leq 2l \quad (2)$$

The periodic conditions are as follows:

$$y(2l) = y(0), \\ y'(2l) = y'(0), \quad (3)$$

Here, ε - the small parameter is given, $\alpha \in (1,2)$, $y(x)$ is the unknown function. First, let's discretize the problem (2)-(3).

Discretization:

In equation (2), if we write the definition of the fractional-order derivative in the Riemann-Liouville sense and discretize it, we get [3, 5, 6, 8-11]:

$$W_{i+1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} W_i + \varepsilon (A_i W_i + U_i), \quad (4) \\ i = \overline{1, 2, \dots}$$

After discretization, the periodicity condi-

tion (3) will be as follows:

$$W_0 = W_{2n} \quad (5)$$

Here $W_n = \begin{pmatrix} y_{2n} \\ y_{2n+1} \end{pmatrix}$ column vector containing the solution to the equation (2),

$$A_i = \begin{pmatrix} A_{11}^{(i-1)} & A_{12}^{(i-1)} \\ A_{21}^{(i-1)} & A_{22}^{(i-1)} \end{pmatrix}$$

$$A_{11}^{(i-1)}(a, i, h) = -h \left[a \frac{(x_{2i-2} - x_{2i-4})^{1-\alpha}}{(1-\alpha)!} + b(x_{2i-2} - x_{2i-4}) \right] + \\ + 2h \left[a \frac{(x_{2i-2} - x_{2i-3})^{1-\alpha}}{(1-\alpha)!} + b(x_{2i-2} - x_{2i-3}) \right]$$

$$A_{12}^{(i-1)}(a, i, h) = -h \left[a \frac{(x_{2i-2} - x_{2i-3})^{1-\alpha}}{(1-\alpha)!} + b(x_{2i-2} - x_{2i-3}) \right]$$

$$A_{21}^{(i-1)}(a, i, h) = -2h \left[a \frac{(x_{2i-2} - x_{2i-4})^{1-\alpha}}{(1-\alpha)!} + b(x_{2i-2} - x_{2i-4}) \right] + \\ + 4h \left[a \frac{(x_{2i-2} - x_{2i-3})^{1-\alpha}}{(1-\alpha)!} + b(x_{2i-2} - x_{2i-3}) \right] - \\ - h \left[a \frac{(x_{2i-1} - x_{2i-4})^{1-\alpha}}{(1-\alpha)!} + b(x_{2i-1} - x_{2i-4}) \right] + \\ + 2h \left[a \frac{(x_{2i-1} - x_{2i-3})^{1-\alpha}}{(1-\alpha)!} + b(x_{2i-1} - x_{2i-3}) \right]$$

$$A_{22}^{(i-1)}(a, i, h) = -2h \left[a \frac{(x_{2i-2} - x_{2i-3})^{1-\alpha}}{(1-\alpha)!} + b(x_{2i-2} - x_{2i-3}) \right] -$$

$$-h \left[a \frac{(x_{2i-1} - x_{2i-3})^{1-\alpha}}{(1-\alpha)!} + b(x_{2i-1} - x_{2i-3}) \right]$$

$$U_i = \begin{pmatrix} \tilde{f}_{i1} \\ \tilde{f}_{i2} \end{pmatrix}, \tilde{f}_{i1} = h^2 f_{2i-2}, \tilde{f}_{i2} = 2h^2 f_{2i-2} + h^2 f_{2i-1}$$

Problem formulation:

Let's construct the functional as follows:

$$J = \sum_{i=0}^{\infty} (W_i' R W_i + U_i' C U_i) \longrightarrow \min \quad (6)$$

Here:

$R = R' \geq 0, C = C' > 0$ these are the weight matrices and the strikemark indicates the transpose operation.

Choose a control strategy (regulator equation)

$$U_i = K_i W_i \quad (7)$$

so as to minimize the quadratic quality criterion (6) and (4)+(7) will be asymptotically stable systems.

Solution method:

It should be noted that an analogous optimal control problem for (2)-(3) has been solved with the periodicity condition taken into account [2]. Let's write the Euler-Lagrange equations. [1]:

$$\begin{bmatrix} W_{i+1} \\ \lambda_i \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} + \varepsilon A_i & -\frac{1}{2} C^{-1} \\ R & \left(\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} + \varepsilon A_i \right)' \end{bmatrix} \begin{bmatrix} W_i \\ \lambda_{i+1} \end{bmatrix} \quad (8)$$

With periodical conditions:

$$\begin{pmatrix} -2 \left(\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} + \varepsilon A_i \right)^{-1} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} W_0 \\ \lambda_1 \end{pmatrix} + \begin{pmatrix} 2R & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} W_{2n} \\ \lambda_{2n} \end{pmatrix} = \begin{pmatrix} - \left(\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} + \varepsilon A_i \right)^{-1} \\ 2\alpha^T \end{pmatrix}$$

Let's search for the solution in the form

$\lambda_i = S(i) W_i$ [1], then:

$$S(i) = \varphi'(i) \left[S(i+1) - S(i+1) (C + S(i+1))^{-1} S(i+1) \right] \varphi(i) + R(i) \quad (9)$$

Here:

$$\varphi(i) = \varphi_0(i) + \varepsilon \varphi_1(i), R(i) = R_0(i) + \varepsilon R_1(i)$$

Thus, to determine the control law, it is necessary to construct a sequence of matrices $S(i)$ satisfying (9). In these problems (for a finite time interval ($i = 1, 2, 3, \dots, 2n$), the value of the matrix $S(2n)$ is given at the right end of the interval. For the case of infinite time interval ($i = 1, 2, 3, \dots$), the definition of the matrix $S(i)$ for any value of i is an independent problem.

$$S(i) = \left[\varphi(i, p)_0 + \varepsilon \varphi(i, p)_1 \right]' \left[E + S(p+i) \left[M(i, p)_0 + \varepsilon M(i, p)_1 \right] \right]^{-1} \times$$

$$\times S(p+i) \left[\varphi(i, p)_0 + \varepsilon \varphi(i, p)_1 \right] + \left[R(i, p)_0 + \varepsilon R(i, p)_1 \right] \quad (10)$$

Here:

$$\varphi_0(i, p) = (\varphi_0(p-1))' [E + M(i, p-1)R]^{-1} \varphi(i, p-1)$$

$$\varphi_1(i, p) = [E + M(i, p-1)R]^{-1} \varphi(i, p-1)$$

$$R_0(i, p) = R(i, p-1) + \varphi_0'(i, p-1) R [E + M(i, p-1)R]^{-1} \varphi_0(i, p-1)$$

$$R_1(i, p) = \varphi_1'(i, p-1) R [E + M(i, p-1)R]^{-1} \varphi_0(i, p-1) +$$

$$+ \varphi_1'(i, p-1) R [E + M(i, p-1)R]^{-1} \varphi_0(i, p-1)$$

$$M_0(i, p) = (\varphi_0(p-1))' [E + M(i, p-1)R]^{-1} M(i, p-1) (\varphi_0(p-1))$$

$$M_1(i, p) = (\varphi_1(p-1))' [E + M(i, p-1)R]^{-1} M(i, p-1) (\varphi_1(p-1)) +$$

$$+ (\varphi_1'(p-1)) [E + M(i, p-1)R]^{-1} M(i, p-1) (\varphi_0(p-1))$$

For each $i = 1, 2, \dots, 2n$ in (10) we get:

$$S(0) = \left[\varphi(0, p)_0 + \varepsilon \varphi(0, p)_1 \right]' \left[E + S(p+0) \left[M(0, p)_0 + \varepsilon M(0, p)_1 \right] \right]^{-1} \times$$

$$\times S(p+0) \left[\varphi(0, p)_0 + \varepsilon \varphi(0, p)_1 \right] + \left[R(0, p)_0 + \varepsilon R(0, p)_1 \right]$$

$$S(1) = \left[\varphi(1, p)_0 + \varepsilon \varphi(1, p)_1 \right]' \left[E + S(p+1) \left[M(1, p)_0 + \varepsilon M(1, p)_1 \right] \right]^{-1} \times$$

$$\times S(p+1) \left[\varphi(1, p)_0 + \varepsilon \varphi(1, p)_1 \right] + \left[R(1, p)_0 + \varepsilon R(1, p)_1 \right]$$

.....

$$S(2n) = \left[\varphi(2n, p)_0 + \varepsilon \varphi(2n, p)_1 \right]' \left[E + S(p+2n) \left[M(2n, p)_0 + \varepsilon M(2n, p)_1 \right] \right]^{-1} \times$$

$$\times S(p+2n) \left[\varphi(2n, p)_0 + \varepsilon \varphi(2n, p)_1 \right] + \left[R(2n, p)_0 + \varepsilon R(2n, p)_1 \right]$$

Note that due to the periodicity of the matrices in the condition of the problem, control strategy will not change if the beginning is shifted by p ($i = p, p+1, \dots$). Therefore, the initial sequence of matrices must also satisfy the periodicity condition:

$$S(i) = S(i+p). \quad (11)$$

If we take this into account in the formula (10) and write the separation ε for each $S(i)$ in the form $S_0(i) + \varepsilon S_1(i)$, and simplify in the first approximation, we will obtain the follo-

wing:

$$S_0(i) + \varepsilon S_1(i) = \varphi_0'(i, p) [E + S_0(i) M_0(i, p)]^{-1} S_0(i) \varphi_0(i, p) +$$

$$\left[\begin{aligned} &\varphi_0'(i, p) \left[(E + S_0(i) M_0(i, p))^{-1} [S_0(i) M_0(i, p) + S_0(i) M_1(i, p)]^{-1} [E + S_0(i) M_0(i, p)]^{-1} \right] \times \\ &\times S_0(i) \varphi_0(i, p) + \varphi_0'(i, p) [E + S_0(i) M_0(i, p)]^{-1} S_1(i) \varphi_0(i, p) + \\ &+ \varepsilon \left[\begin{aligned} &+ \varphi_0'(i, p) [E + S_0(i) M_0(i, p)]^{-1} S_1(i) \varphi_0(i, p) + \\ &+ \varphi_0'(i, p) [E + S_0(i) M_0(i, p)]^{-1} S_0(i) \varphi_1(i, p) + \\ &+ \varphi_1'(i, p) [E + S_0(i) M_0(i, p)]^{-1} S_0(i) \varphi_0(i, p) \end{aligned} \right] \end{aligned} \right] \quad (12)$$

We will obtain. From equation (10), we will get that:

$$S_0(i) = \varphi_0'(i, p) [E + S_0(i) M_0(i, p)]^{-1} S_0(i) \varphi_0(i, p) + R(i, p)_0 \quad (13)$$

$$S_1(i) = \varphi_0'(i, p) \left[(E + S_0(i) M_0(i, p))^{-1} [S_0(i) M_0(i, p) + S_0(i) M_1(i, p)]^{-1} [E + S_0(i) M_0(i, p)]^{-1} \right] S_0(i) \varphi_0(i, p) +$$

$$\varphi_0'(i, p) [E + S_0(i) M_0(i, p)]^{-1} S_1(i) \varphi_0(i, p) + \varphi_0'(i, p) [E + S_0(i) M_0(i, p)]^{-1} S_0(i) \varphi_1(i, p) + \varphi_1'(i, p) [E + S_0(i) M_0(i, p)]^{-1} S_0(i) \varphi_0(i, p) + R(i, p)_1 \quad (14)$$

Equation (13), as seen, is a discrete algebraic Riccati equation, and we know the method to solve it. Equation (14) can be solved using the Gradient method.

Thus, we will obtain the solution to the problem as follows:

$$W_{i+1} = (W_{i+1})_0 + \varepsilon (W_{i+1})_1$$

$$i = 0, 1, 2, \dots \quad (15)$$

Here:

$$(W_{i+1})_0 = \left[E + C^{-1} (S_i)_0 \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \right]^{-1} * (W_i)_0 \quad (16)$$

$$(W_{i+1})_1 = \left[E + C^{-1} (S_i)_0 \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \right]^{-1} (W_i)_1 +$$

$$+ \left[\begin{aligned} &(E + C^{-1} (S_i)_0)^{-1} (E + C^{-1} (S_i)_0)^* \\ &+ (E + C^{-1} (S_i)_0)^{-1} \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \end{aligned} \right] (W_i)_0 \quad (17)$$

The controller will be defined as [1]:

$$U_i = (U_i)_0 + \varepsilon (U_i)_1$$

Here:

$$(U_i)_0 = -\frac{1}{2} \left(C + \frac{1}{2} (S_{i+1})_0 \right)^{-1} (S_{i+1})_0 (W_i)_0$$

$$(U_i)_1 = -\frac{1}{2} \left\{ \left[\begin{aligned} &\left(C + \frac{1}{2} (S_{i+1})_0 \right)^{-1} (S_{i+1})_0 \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} (W_i)_1 + \\ &\left[\left(C + \frac{1}{2} (S_{i+1})_0 \right)^{-1} \left(C + \frac{1}{2} (S_{i+1})_0 \right)^{-1} \left(C + \frac{1}{2} (S_{i+1})_0 \right)^{-1} \right] (S_{i+1})_0 + \\ &\left[\left(C + \frac{1}{2} (S_{i+1})_0 \right)^{-1} \right] (S_{i+1})_0 \tilde{A}^* + \left[\left(C + \frac{1}{2} (S_{i+1})_0 \right)^{-1} \right] (S_{i+1})_1 \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \end{aligned} \right] (W_i)_0 \right\} \quad (18)$$

According to in (16) $E + C^{-1} (S_i)_0 \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$ and

$$\text{in (17)} \left[\begin{aligned} &(E + C^{-1} (S_i)_0)^{-1} (E + C^{-1} (S_i)_0)^* \\ &+ (E + C^{-1} (S_i)_0)^{-1} \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \end{aligned} \right]$$

if the elements of the matrix and the corresponding matrix are located within the unit circle, the (2)-(3) system will be asymptotically stable.

Conclusion

In the end, the issue of establishing the stability of the optimal controller in the periodic optimization problem of discrete systems with liquid dampers for the asymptotic case has been examined and a solution algorithm has been proposed. In the solution process, separation with respect to the small parameter has been obtained, resulting in two equations: the first is a discrete algebraic Riccati equation, for which we know the solution method, and the second is proposed to be solved using the gradient method.

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DISKRET HAL ÜÇÜN MAYE DEMPFERLİ RƏQSI SİSTEMLƏRDƏ PERİODİK OPTİMİZASIYA MƏSƏLƏSİNDƏ ASİMPOTOTİK HƏLLİN QURULMASI ÜSULU

A.F.Rəsulzadə, N.İ.Vəliyeva

Ştanqlı nasos qurğusu ilə neftçıxarma zamanı idarəetmə və stabilizasiya sistemlərinin qurulması məsələsi kəsr tərtib daxil olan kiçik parametrlı diferensial tənliklər ilə tədqiq olunur. Onlar üçün optimal idarəetmə məsələsi qoyulur və qoyulmuş məsələ üçün birinci yaxınlaşmada kiçik parametrlı stabilizasiya məsələsi həll olunur.

Açar sözlər: Ştanqlı nasos qurğusu, asimptotika, diskretləşdirmə, Eyler-Laqrانج tənliyi, cəbri diskret Rikkatti tənliyi, idarəetmə, dayanıqlılıq, kiçik parametrlı

МЕТОД ПОСТРОЕНИЯ АСИМПТОТИЧЕСКОГО РЕШЕНИЯ ЗАДАЧИ ПЕРИОДИЧЕСКОЙ ОПТИМИЗАЦИИ ДЛЯ ДИСКРЕТНЫХ СИСТЕМ С ЖИДКОСТНЫМ ДЕМПФЕРОМ

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Вопрос построения систем управления и стабилизации при добыче нефти с использованием штангового насосного устройства исследуется с применением дифференциальных уравнений с малым параметром, входящих в дробные формы. Для них формулируется задача оптимального управления, и на первом приближении решается задача стабилизации с малыми параметрами для данной задачи.

Ключевые слова: штанговый насос, асимптотика, дискретизация, уравнение Эйлера-Лагранжа, алгебраическое дискретное уравнение Риккати, управление, устойчивость, малый параметр