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ON THE SOLVABILITY OF THE HOMOGENEOUS RIEMANN PROBLEM IN THE MORREY-HARDY CLASSES

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The homogeneous Riemann boundary value problem for analytic functions in Morrey-Hardy classes is considered. In this case, some properties of functions from these classes are studied and the possibility of representing these functions in terms of an integral of Cauchy type is established. The specificity of these problems lies in the fact that the Morrey space $L^{p,\alpha}$ is not separable. Therefore, we consider a subspace $M^{p,\alpha} \subset L^{p,\alpha}$, in which the continuous functions are dense. From the point of view of the approximation theory, the space $M^{p,\alpha}$ is of special interest. The corresponding Morrey-Hardy classes generated by the space $M^{p,\alpha}$ are defined and the solvability problems of Riemann problems in these classes are studied.

Keywords: Morrey space, Morrey-Hardy classes, Riemann problem, solvability

Introduction

The theory of Riemann boundary value problems for analytic functions has a deep and long history. This problem was first posed by B.Riemann. Then D.Hilbert drew attention to similar problems and formulated the statement, so-called now, of the Riemann-Hilbert problem. Later this direction in connection with applications in the solution of problems of mechanics and mathematical physics was essentially developed in the works of well-known mathematicians and the theory of these problems is illuminated in well-known monographs. It should be noted that the methods of this theory are also applied in other fields of mathematics (for example, in approximation theory, in the spectral theory of differential operators, etc.).

Recently, in connection with applications in problems of mechanics and pure mathematics, interest in nonstandard function spaces (e.g. Lebesgue space with variable summability exponent, Morrey space, "grand" Lebesgue spaces, etc.) has greatly increased. Review articles and monographs are devoted to this direction (see e.g. [1-3]). Different problems of mathematical analysis (e.g. the boundedness problems of various integral operators such as singular operators, Riesz potentials and etc., direct and inverse problems of approxi-

mation theory with respect to Faber polynomials and others) are studied in similar spaces. Along with this, Riemann-Hilbert problems in these spaces have also been studied in various formulations (see, for example, [4-8]), and many problems remain unresolved until now.

In this work the homogeneous Riemann boundary value problem for analytic functions in Morrey-Hardy classes is considered. In this case, some properties of functions from these classes are studied and the possibility of representing these functions in terms of an integral of Cauchy type is established. The specificity of these problems lies in the fact that the Morrey space $L^{p,\alpha}$ is not separable. Therefore, we consider a subspace $M^{p,\alpha} \subset L^{p,\alpha}$, in which the continuous functions are dense. From the point of view of the approximation theory, the space $M^{p,\alpha}$ is of special interest. The corresponding Morrey-Hardy classes generated by the space $M^{p,\alpha}$ are defined and the solvability problems of the homogeneous Riemann problems in these classes are studied.

Needful information

Let us give some notation and facts, which we will use when obtaining the main results. First we define the Morrey space on the

unit circle $\gamma = \{z \in \mathbb{C} : |z| = 1\}$ on the complex plane $\mathbb{C}$. Further by $\omega = \text{int } \gamma$ we will denote the unit ball in $\mathbb{C}$. By $L_0(-\pi, \pi)$ denote the linear space of all Lebesgue measurable functions on $(-\pi, \pi)$.

Denote by $L^{p,\alpha}(\gamma)$, $1 \leq p < +\infty$, $0 \leq \alpha \leq 1$, the normed space of all measurable functions $f(\cdot)$ on γ with a finite norm

$$\|f\|_{L^{p,\alpha}(\gamma)} = \sup_B \left(|B \cap \gamma|^{\alpha-1} \int_{B \cap \gamma} |f(\xi)|^p |d\xi| \right)^{1/p} < +\infty,$$

($|B \cap \gamma|_\gamma$ – is a linear measure of $B \cap \gamma$), where $\sup$ is taken over all balls centered on γ and with an arbitrary positive radius. With respect to this norm $L^{p,\alpha}(\gamma)$ is a Banach space.

Using the results of [9], the following lemma is easily proved.

Lemma 2.1. Let $\{\tau_k\}_{k=1, \overline{m}} \subset \gamma$ different points. Then the finite product

$$\omega(\tau) = \prod_{k=1}^m |\tau - \tau_k|^{\alpha_k}, \quad \tau \in \gamma,$$

belongs to the space $L^{p,\alpha}$, $1 \leq p < +\infty$, $0 < \alpha < 1$, if and only if the inequalities $\alpha_k \geq -\frac{\alpha}{p}$, $\forall k = \overline{1, m}$ are fulfilled.

Morrey-Hardy space

Let us give some necessary information from the theory of Hardy spaces.

Define the class Morrey-Hardy $H_+^{p,\alpha}$, $1 \leq p < +\infty$, $0 \leq \alpha \leq 1$, of analytic functions $f(\cdot)$ inside the unit disc ω with a norm

$$\|f\|_{H_+^{p,\alpha}} = \sup_{0 < r < 1} \|f_r(\cdot)\|_{p,\alpha},$$

where $f_r(t) = f(re^{it})$. It is easy to see that the inclusion $H_+^{p,\alpha} \subset H_1^+$, $1 \leq p < +\infty$, holds true, where H_1^+ is an ordinary Hardy class.

Consider the space $H_+^{p,\alpha}$. Denote by $L_+^{p,\alpha}$, the subspace $L^{p,\alpha}$, generated by the restrictions of functions from $H_+^{p,\alpha}$ on γ , i.e. $L_+^{p,\alpha} = H_+^{p,\alpha} / \gamma$.

The spaces $H_+^{p,\alpha}$ and $L_+^{p,\alpha}$ are isomorphic and the restriction operator $J : H_+^{p,\alpha} \leftrightarrow L_+^{p,\alpha}$; $(Jf)(\tau) = f^+(\tau)$, $\tau \in \gamma$, $\forall f \in H_+^{p,\alpha}$, carries out the corresponding isomorphism. Put $M_+^{p,\alpha} = M^{p,\alpha} \cap L_+^{p,\alpha}$, with a norm $\|\cdot\|_{p,\alpha}$. It is clear that $M_+^{p,\alpha}$ is a subspace of $M^{p,\alpha}$ (since $M^{p,\alpha}$ and $L_+^{p,\alpha}$ both are closed subspaces of $L^{p,\alpha}$). Let $MH_+^{p,\alpha} = J^{-1}(M_+^{p,\alpha})$. Obviously, $MH_+^{p,\alpha}$ is a subspace of $H_+^{p,\alpha}$. From the above considerations it follows that for $\forall f \in H_+^{p,\alpha}$ the norm $\|f\|_{H_+^{p,\alpha}}$ can also determine by $\|f\|_{H_+^{p,\alpha}} = \|f^+\|_{p,\alpha}$, where f^+ – is the non-tangential boundary values of f along γ .

Similarly to the case $MH_+^{p,\alpha}$, we define the class ${}_mMH_-^{p,\alpha}$. In other words, ${}_mMH_-^{p,\alpha}$ is the subspace of functions from ${}_mH_-^{p,\alpha}$, which shifts on γ are continuous with respect to the norm $\|\cdot\|_{p,\alpha}$.

Let us consider the weighted variants of the above spaces. First we define a weighted space $M_\rho^{p,\alpha}$ with some weight function. Consider the weighted space $L_\rho^{p,\alpha}$ and let $f \in L_\rho^{p,\alpha}$ – be some function. If necessary, we consider that the function $f(\cdot)$ outside of the segment $[-\pi, \pi]$ is parity continued, i.e. (the extended function will also be denoted by $f(\cdot)$)

$$f(x) = \begin{cases} f(-2\pi - x), & x \in [-3\pi, -\pi), \\ f(2\pi - x), & x \in (\pi, 3\pi]. \end{cases}$$

Put

$$M_\rho = \{f \in L_\rho^{p,\alpha} : \|f(\cdot + \delta) - f(\cdot)\|_{p,\alpha;\rho} \rightarrow 0, \delta \rightarrow 0\}.$$

It is quite obvious that M_ρ is a linear subspace of $L_\rho^{p,\alpha}$. Denote by $M_\rho^{p,\alpha}$ denote the closure of M_ρ in $L_\rho^{p,\alpha}$, i.e. $M_\rho^{p,\alpha}$ is a subspace of $L_\rho^{p,\alpha}$.

On a subspace of the dual Morrey-Lebesgue space

Since we do not know the characterization of the dual space to the Morrey-Lebesgue space $L^{p,\alpha}$, in the form of a function space, therefore it is sufficient to consider a subspace of the space $(L^{p,\alpha})^*$, denoted as $(L^{p,\alpha})'$ and defined by the relation

$$(L^{p,\alpha})' = \left\{ g \in L_0(-\pi, \pi) : \sup_{f \in S_{p,\alpha}} \|fg\|_{L_1(-\pi, \pi)} < +\infty \right\},$$

with a norm

$$\|g\|_{(L^{p,\alpha})'} = \sup_{f \in S_{p,\alpha}} \|fg\|_{L_1}, \quad (4.1)$$

where $S_{p,\alpha} = \{f \in L^{p,\alpha} : \|f\|_{p,\alpha} = 1\}$ – is a unit sphere in $L^{p,\alpha}$.

The following theorem is true.

Theorem 4.1.

$(L^{p,\alpha})'$, $1 \leq p < +\infty$, $0 \leq \alpha \leq 1$ is a Banach space with respect to the norm (4.1).

It is easy to prove the following lemma.

Lemma 4.1. The finite product

$$\nu(t) = \prod_{k=1}^m |t - t_k|^{\beta_k},$$

$$\{t_k\}_{k=1, \overline{m}} \subset [-\pi, \pi], \quad t_i \neq t_j, \quad i \neq j,$$

belongs to the space $(L^{p,\alpha})'$, $1 \leq p < +\infty$, $0 < \alpha \leq 1$,

if and only if $\beta_k \in \left(-1 + \frac{\alpha}{p}, +\infty\right)$, $\forall k = \overline{1, m}$.

The general solution of the homogeneous Riemann problem in Morrey-Hardy classes

Consider the following homogeneous Riemann problem

$$\left. \begin{aligned} F^+(\tau) - G(\tau)F^-(\tau) &= 0, \quad \tau \in \gamma, \\ F^+(\cdot) &\in H_+^{p,\alpha}; F^-(\cdot) \in H_-^{p,\alpha}, \end{aligned} \right\} \quad (5.1)$$

$$1 < p < +\infty, \quad 0 < \alpha \leq 1,$$

where $G(e^{it}) = |G(e^{it})|e^{i\theta(t)}$, $t \in [-\pi, \pi]$ – is a coefficient of the problem. By solution of the problem (5.1) we mean a pair of analytic functions $(F^+; F^-) \in H_+^{p,\alpha} \times_m H_-^{p,\alpha}$, whose non-tangential boundary values $F^+(\tau)$ inside ω and $F^-(\tau)$ outside ω on γ satisfy the relation (5.1) a.e. on γ . This problem will be solved according to the scheme proposed in the monograph of I.I. Danilyuk [10]. Let us introduce the following piecewise analytic functions on the complex plane with a cut γ :

$$X_1(z) \equiv \exp \left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln |G(e^{it})| \frac{e^{it} + z}{e^{it} - z} dt \right\},$$

$$X_2(z) \equiv \exp \left\{ \frac{i}{4\pi} \int_{-\pi}^{\pi} \theta(t) \frac{e^{it} + z}{e^{it} - z} dt \right\}, \quad z \notin \gamma.$$

Let

$$Z_k(z) = \begin{cases} X_k(z), & |z| < 1, \\ (X_k(z))^{-1}, & |z| > 1, \end{cases}$$

and assume $Z(z) = Z_1(z)Z_2(z)$, $z \notin \gamma$.

We will assume that the coefficient of the problem (5.1) $G(\cdot)$ satisfies the following conditions:

$$i) G^{\pm 1}(\cdot) \in L_\infty(-\pi, \pi);$$

ii) $\theta(t) = \arg G(e^{it})$ – is a piecewise Holder function on $[-\pi, \pi]$ and $h_k = \theta(s_k + 0) - \theta(s_k - 0)$, $k = \overline{1, r} - \pi$ are jumps of this function at the discontinuity points $\{s_k\}_1^r : -\pi < s_1 < \dots < s_r < \pi$.

The following theorem is true.

Theorem 5.1. Let the coefficient $G(\cdot)$ of the problem (5.1) satisfy the conditions i), ii) and the jumps $\{h_k\}_0^r$ of the function $\theta(t) = \arg G(e^{it})$ in $[-\pi, \pi]$ satisfy the inequalities

$$-1 + \frac{\alpha}{p} < \frac{h_k}{2\pi} \leq \frac{\alpha}{p}, \quad k = \overline{0, r}, \quad (5.2)$$

where $h_0 = \theta(-\pi) - \theta(\pi)$. Then

$\alpha)$ for $m \geq 0$ the problem (5.1) has a general solution of the form

$$F(z) \equiv Z(z)P_k(z), \quad (5.3)$$

where $Z(z)$ – is a canonical solution of this problem, $P_k(z)$ – is an arbitrary polynomial of order $k \leq m$;

$\beta)$ for $m < 0$ the problem (5.1) has only trivial solution.

The following theorem is true.

Theorem 5.2. Let the coefficient $G(\cdot)$ satisfy the conditions i), ii) and the jumps $\{h_k\}_0^r$ of the function $\theta(\cdot)$ satisfy the inequalities

$$-1 + \frac{\alpha}{p} < \frac{h_k}{2\pi} < \frac{\alpha}{p}, \quad k = \overline{0, r}. \quad (5.4)$$

Then

$\alpha)$ for $m \geq 0$ the problem (5.1) has a general solution of the form (5.3) in Morrey-Hardy classes $MH_+^{p,\alpha} \times_m MH_-^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha \leq 1$;

$\beta)$ for $m < 0$ the problem (5.1) has only trivial solution.

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BİRCİNS RİMAN MƏSƏLƏSİNİN MORRI-HARDİ SİNİFLƏRİNDƏ HƏLLÖLÜNANLIĞI HAQQINDA

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İşdə analitik funksiyalar üçün Morri-Hardi siniflərində bircins Riman məsələsinə baxılır. Eyni zamanda bu siniflərdən olan funksiyaların bəzi xassələri öyrənilmiş və bu funksiyaların Koşi tip integral vasitəsilə ifadəsinin mümkünlüyü isbat olunmuşdur. Məsələnin məxsusiliyi ondan ibarətdir ki, $L^{p,\alpha}$ Morri fəzası separabel fəza deyil. Bu səbəbdən kəsilməz funksiyaların sıx olduğu $M^{p,\alpha} \subset L^{p,\alpha}$ alt fəzaya baxılır. Approksimasiya nəzəriyyəsi nöqtəyi-nəzərdən $M^{p,\alpha}$ fəzası xüsusi maraq doğurur. $M^{p,\alpha}$ fəzasının doğurduğu uyğun Morri-Hardi sinifləri təyin olunur və bu siniflərdə Riman məsələsinin həllölanlılığı öyrənilir.

Açar sözlər: Morri fəzası, Morri-Hardi sinifləri, Riman məsələsi, həllölanlılıq

О РАЗРЕШИМОСТИ ОДНОРОДНОЙ ЗАДАЧИ РИМАНА В КЛАССАХ МОРРИ-ХАРДИ

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В работе рассматривается однородная краевая задача Римана в классах Морри-Харди для аналитических функций. При этом изучаются некоторые свойства функций из этих классов и устанавливается возможность представления через интеграл типа Коши этих функций. Специфичность этих задач заключается в том, что пространство Морри $L^{p,\alpha}$ не является сепарабельным. Поэтому рассматривается подпространство $M^{p,\alpha} \subset L^{p,\alpha}$, в котором непрерывные функции плотны. С точки зрения теории аппроксимации пространство $M^{p,\alpha}$ представляет особый интерес. Определяются соответствующие классы Морри-Харди, порожденные пространством $M^{p,\alpha}$ и изучаются вопросы разрешимости задач Римана в этих классах.

Ключевые слова: пространство Морри, классы Морри-Харди, задача Римана, разрешимость