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EIGENFUNCTION EXPANSIONS FOR THE SCHRÖDINGER OPERATOR WITH A SQUARE POTENTIAL

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The Schrödinger operator $L = -\frac{d^2}{dx^2} + \theta(x)x^2$ is considered on the entire axis, where $\theta(x)$ is Heaviside function. The direct scattering problem for the operator L is studied. The formula is obtained for the expansion in terms of eigenfunctions of the continuous spectrum of the operator L .

Keywords: harmonic oscillator, Schrödinger operator, spectrum, eigenfunction, Weber function

Introduction

Let us consider the one-dimensional Schrödinger equation

$$-y'' + \theta(x)x^2 y = \lambda y, \quad -\infty < x < \infty, \quad \lambda \in \mathbb{C}, \quad (1)$$

where $\theta(x)$ is Heaviside function. Differential equation (1) defines in space $L_2(-\infty, +\infty)$ a self-adjoint operator L , which can be obtained by closure of symmetric operator defined by equation (1.1) on twice continuously differentiable finite functions. In this paper, the direct scattering problem for the operator L is studied. A formula is obtained for the expansion in terms of eigenfunctions of the continuous spectrum of the operator L .

Note that various spectral problems for a harmonic oscillator, i.e. for the Schrodinger equation with a square potential, in various contexts has been studied by many authors (see. [1]–[5] and bibliography in them).

Scattering problem. It is known [6] that the equation

$$-y'' + x^2 y = \lambda y, \quad -\infty < x < +\infty,$$

has a pair of linearly independent solutions $D_{\frac{\lambda-1}{2}}(\sqrt{2}x)$ and $D_{\frac{\lambda-1}{2}}(-\sqrt{2}x)$, where $D_\nu(x)$ is

Weber function. Let us denote by G the com-

plex λ - plane with a cut along the positive semi-axis. Let us consider a function $\sqrt{\lambda}$ in the plane G , choosing a regular branch of the radical such that $\sqrt{\lambda + i0} > 0$ for $\lambda > 0$.

Theorem 1. For each complex value λ , the equation (1) has solutions $\psi_\pm(x, \lambda)$, expressed as follows:

$$\psi_+(x, \lambda) = \begin{cases} D_{\frac{\lambda-1}{2}}(\sqrt{2}x), & x \geq 0, \\ \frac{1}{2} \left[D_{\frac{\lambda-1}{2}}(0) - i \sqrt{\frac{2}{\lambda}} D'_{\frac{\lambda-1}{2}}(0) \right] e^{i\sqrt{\lambda}x} + \\ + \frac{1}{2} \left[D_{\frac{\lambda-1}{2}}(0) + i \sqrt{\frac{2}{\lambda}} D'_{\frac{\lambda-1}{2}}(0) \right] e^{-i\sqrt{\lambda}x}, & x < 0, \end{cases} \quad (2)$$

$$\psi_-(x, \lambda) = \begin{cases} \frac{1}{2} \left[D_{\frac{\lambda-1}{2}}^{-1}(0) - i \sqrt{\frac{2}{\lambda}} \left(D'_{\frac{\lambda-1}{2}}(0) \right)^{-1} \right] D_{\frac{\lambda-1}{2}}(\sqrt{2}x) + \\ + \frac{1}{2} \left[D_{\frac{\lambda-1}{2}}^{-1}(0) + i \sqrt{\frac{2}{\lambda}} \left(D'_{\frac{\lambda-1}{2}}(0) \right)^{-1} \right] D_{\frac{\lambda-1}{2}}(-\sqrt{2}x), & x \geq 0, \\ e^{-i\sqrt{\lambda}x}, & x < 0. \end{cases} \quad (3)$$

Note that for $\lambda \in (0, +\infty)$, the function $\overline{\psi_-(x, \lambda)}$ is also a solution to equation (1). Moreover, for $\lambda \in (0, +\infty)$, the solutions $\psi_-(x, \lambda)$ and $\overline{\psi_-(x, \lambda)}$ are linearly independent, since their Wronskian is equal to $2i\sqrt{\lambda}$. Taking into account that for $\lambda \in (-\infty, +\infty)$ the function $\psi_+(x, \lambda)$ takes real values, we obtain the expansion

$$\psi_+(x, \lambda) = a(\lambda) \overline{\psi_-(x, \lambda)} + \overline{a(\lambda)} \psi_-(x, \lambda),$$

$$\lambda \in (0, +\infty), \quad (4)$$

where the function $a(\lambda)$ is defined by formula

$$a(\lambda) = (2i\sqrt{\lambda})^{-1} \left[\sqrt{2} D'_{\frac{\lambda-1}{2}}(0) + i\sqrt{\lambda} D_{\frac{\lambda-1}{2}}(0) \right].$$

Functions $a^{-1}(\lambda) \psi_+(x, \lambda)$ and $r(\lambda) = a^{-1}(\lambda) \overline{a(\lambda)}$ are called the solution to the scattering problem and the reflection coefficient for equation (1) respectively.

Decomposition in terms of eigenfunctions. Using solutions (2), (3), it is proven by the standard method (see. [5]) that for $\lambda \notin [0, +\infty)$, integral operator R_λ is defined in space $L_2(-\infty, +\infty)$ by the formula

$$(R_\lambda f)(x) = \int_{-\infty}^{+\infty} R(x, t, \lambda) f(t) dt,$$

where

$$R(x, t, \lambda) = \frac{1}{2i\sqrt{\lambda}a(\lambda)} \psi_+(x, \lambda) \overline{\psi_-(x, \lambda)} \theta(x-t) + \psi_-(x, \lambda) \overline{\psi_+(t, \lambda)} \theta(t-x)$$

is the resolvent of the operator L . Furthermore, the operator L has continuous operator, which fills the positive semi-axis $[0, +\infty)$. Explicit formula for the resolvent R_λ leads to the theorem of expansion in terms of eigenfunctions of the operator L . Namely, we denote by $E(\Delta)$, where Δ runs the Borel subsets in $(-\infty, +\infty)$, decomposition of the identity of a self-adjoint operator L (see. [7]). In the absence of a point spectrum, the following formula is valid:

$$E(\Delta) = \lim_{\varepsilon \rightarrow +0} \frac{1}{2\pi i} \int_{\Delta} (R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)) d\lambda$$

(see. [7]). This formula is sometimes called Stone's formula. In particular, assuming $\Delta = [0, +\infty)$, for the operator L we get

$$I = \lim_{\varepsilon \rightarrow +0} \frac{1}{2\pi i} \int_0^{+\infty} (R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)) d\lambda$$

This formula and relation (4) serve as the basis for the derivation of the decomposition theorem.

Theorem 2. The decomposition formula

$$\frac{1}{4\pi} \int_0^{+\infty} \frac{1}{|a(\lambda)|^2 \sqrt{\lambda + i0}} \psi_+(x, \lambda) \overline{\psi_+(y, \lambda)} d\lambda = \delta(x - y)$$

is valid, where δ is Dirac's delta function..

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KVADRATİK POTENSİALLI ŞREDİNGER OPERATORUNUN MƏXSUSİ FUNKSİYALARI ÜZRƏ AYRILIŞ DÜSTURU

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Bütün oxda $L = -\frac{d^2}{dx^2} + \theta(x)x^2$ Şredinger operatoruna baxılmışdır, burada $\theta(x)$ Hevisayd funksiyasıdır.

L operatoru üçün səpilmənin düz məsələsi öyrənilmişdir. L operatorunun kəsilməz spektrinin məxsusi funksiyaları üzrə ayrılış düsturu çıxarılmışdır.

Açar sözlər: *Harmonik osilyator, Şredinger operatoru, spektr, məxsusi funksiya, Veber funksiyası*

ФОРМУЛА РАЗЛОЖЕНИЯ ПО СОБСТВЕННЫМ ФУНКЦИЯМ ОПЕРАТОРА ШРЕДИНГЕРА С КВАДРАТИЧНЫМ ПОТЕНЦИАЛОМ

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Рассмотрен оператор Шредингера $L = -\frac{d^2}{dx^2} + \theta(x)x^2$ на всей оси, где $\theta(x)$ – функция Хевисайда.

Изучена прямая задача рассеяния для оператора L . Получена формула разложения по собственным функциям непрерывного спектра оператора L .

Ключевые слова: *гармонический осциллятор, оператор Шредингера, спектр, собственная функция, функция Вебера*