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ON THE INVERSE PROBLEM FOR A SCHRODINGER EQUATION WITH AN ADDITIONAL LINEAR POTENTIAL

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The paper considers the Schrodinger equation with an additional linear potential on the entire axis. The transformation operator with the condition at $-\infty$ is constructed. The Gelfand-Levitan integral equation is obtained on the semi axis $(-\infty, x)$.

Keywords: Schrodinger equation, additional linear potential, Airy functions, transformation operator, inverse problem, main equation

1. In [1],[2], the inverse spectral problem for the Schrodinger equation is studied in the form

$$-y'' + xy + q(x)y = \lambda y, \quad -\infty < x < \infty. \quad (1)$$

Moreover, on the basis of the triangular representation of the solution of equation (1), the main integral Gelfand-Levitan equation for the right end is obtained (see [1],[2]). At the same time, there remains open the question (see [3]) about the main equation for the left end, since the absence of this equation does not allow us to draw a conclusion about the behavior of the potential being restored at $-\infty$.

Let us consider equation (1), in which the real potential is continuously differentiated, and satisfies the conditions

$$\int_0^{\infty} |xq(x)|dx < \infty, \quad \int_{-\infty}^0 e^{2|x|^{\frac{3}{2}}} |xq(x)|dx < \infty. \quad (2)$$

In the present paper, the Gelfand-Levitan integral equation is obtained for the left end.

Note that inverse problems for the Schrodinger equation with growing potentials were studied in [4]–[7].

2. It is known [8] that the Airy functions $Ai(x-\lambda)$ and $Bi(x-\lambda)$ are linearly independent solutions of equation (1) for $q(x) = 0$. We also

introduce solutions

$$f_+^0(x, \lambda) = \sqrt{\pi} Ai(x-\lambda), \quad f_-^0(x, \lambda) = \sqrt{\pi} [Ai(x-\lambda) - iBi(x-\lambda)]$$

that for $\text{Im} \lambda > 0$ satisfy [1] the relations

$$f_{\pm}^0(x, \lambda) \in L_2(0, \pm\infty).$$

$$\text{Let} \quad \sigma_i^-(x) = \int_{-\infty}^x |t|^i e^{2|t|^{\frac{3}{2}}} |q(t)| dt, \quad i = 0, 1.$$

Then the following theorem is valid.

Theorem 1. If potential $q(x)$ satisfies condition (2), then equation (1) for all $\lambda, \text{Im} \lambda \geq 0$ has unique solution $f_-(x, \lambda)$ in the form

$$f_-(x, \lambda) = f_0^-(x, \lambda) + \int_{-\infty}^x K^-(x, t) f_0^-(t, \lambda) dt, \quad (3)$$

where $K^-(x, t)$ is real continuous function, and

$$|K^-(x, t)| \leq \frac{1}{2} \sigma^-\left(\frac{x+t}{2}\right) e^{\sigma_1^-(x)}, \quad (4)$$

$$K^-(x, x) = \frac{1}{2} \int_{-\infty}^x q(t) dt. \quad (5)$$

3. We use the transformation operator (3) to derive the Gelfand-Levitan integral equation for the left end. It is known [1],[2] that, under condition (2), equation (1) has a solution that can be represented as

$$f_+(x, \lambda) = f_0^+(x, \lambda) + \int_x^{\infty} K^+(x, t) f_0^+(t, \lambda) dt.$$

For real values λ of the solution $f_{\pm}(x, \lambda)$ are related by the identity

$$\frac{f_+(x, \lambda)}{a(\lambda)} = \overline{f_-(x, \lambda)} + r(\lambda) f_-(x, \lambda).$$

The coefficients $a(\lambda), r(\lambda)$ are determined by the formulas

$$a(\lambda) = \frac{i}{2} W[f_+(x, \lambda), f_-(x, \lambda)], \quad r(\lambda) = \frac{\overline{a(\lambda)}}{a(\lambda)},$$

where $W[f_+(x, \lambda), f_-(x, \lambda)]$ is a Wronskian of solutions $f_+(x, \lambda), f_-(x, \lambda)$. Moreover, under condition (2), it is easy to see (see also [1], [2]) the following asymptotic equalities

$$a(\lambda) = \frac{1}{2} + O\left(|\lambda|^{-\frac{1}{2}}\right), \lambda \rightarrow \infty,$$

$$r(\lambda) - 1 \sim \begin{cases} c_1 \lambda^{-\frac{3}{2}} e^{-\frac{2}{3}i\lambda^{\frac{3}{2}}}, & \lambda \rightarrow +\infty, \\ c_2 |\lambda|^{\frac{1}{4}} e^{-\frac{2}{3}|\lambda|^{\frac{3}{2}}}, & \lambda \rightarrow -\infty. \end{cases}$$

Theorem 2. For each fixed x function $K^-(x, y)$ included in the representation (3) satisfies the integral equation

$$F^-(x, y) + K^-(x, y) + \int_{-\infty}^x K^-(x, t) F^-(t, y) dt = 0, \quad y < x, \quad (6)$$

where the kernel $F^-(x, y)$ is defined by the formula

$$F^-(x, y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \operatorname{Re} \{ (r(\lambda) - 1) f_-^0(x, \lambda) f_-^0(y, \lambda) \} d\lambda. \quad (7)$$

Further, the well-known methods for studying the Gelfand-Levitan equation (see, for example, [6]) make it possible to prove its unique solvability. In this case, formulas (5), (6), (7) make it possible to restore the potential $q(x)$ from the reflection coefficient $r(\lambda)$ and to conclude that the potential increases.

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ƏLAVƏ XƏTTİ POTENSİALA MALİK OLAN ŞREDİNGER TƏNLIYİ ÜÇÜN TƏRS MƏSƏLƏ HAQQINDA

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Məqalədə bütün oxda əlavə xətti potensiala malik olan Şredinger tənliyinə baxılır. Mənfi sonsuzluqda şərt ödəyən çevirmə operatoru qurulmuşdur. $(-\infty, x)$ yarımxunda Qelfand-Levitan tənliyi tapılmışdır.

Açar sözlər: Şredinger tənliyi, əlavə xətti potensial, Eyri funksiyası, çevirmə operatoru, tərs məsələ, əsas tənlik

**ОБ ОБРАТНОЙ ЗАДАЧЕ ДЛЯ УРАВНЕНИЯ ШРЕДИНГЕРА С ДОПОЛНИТЕЛЬНЫМ
ЛИНЕЙНЫМ ПОТЕНЦИАЛОМ****А.Х.Ханмамедов, М.Г.Махмудова**

Рассматривается уравнение Шредингера с дополнительным линейным потенциалом. Построен оператор преобразования с условием на $-\infty$. На полуоси $(-\infty, x)$ найдено уравнение Гельфанда-Левитана.

Ключевые слова: уравнение Шредингера, дополнительный линейный потенциал, функция Эйри, оператор преобразования, обратная задача, основное уравнение