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COMPUTATIONAL ALGORITHM FOR SOLVING THE QUASI-LINEAR EQUATION USING THE RELAXATION METHOD

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In the paper the Cauchy problem for a system of partial differential equations with a large parameter that describe modeling of oil production by gas lift is considered. To solve this problem, a sweep method is used. In this case, the solution of two differential equations is required, one of which is a quasilinear partial differential equation, and the second an ordinary linear differential equation. The study of the problem in this form simplifies the solution method and makes it possible to complete the computer realization of the proposed calculation algorithms.

Keywords: gas lift, hyperbolic equations, differential equations, sweep method, grid method, relaxation method

Introduction

One of the most important stages in oil production is the gas lift method. Various mathematical models have been developed that describe the movement in the process of gas lift [1,2,6] and with their help various problems were set, for example, producing the maximum amount of oil with a minimum gas volume [3,5,8], the determination of the hydraulic resistance coefficient [4] etc. In this paper, we consider the application of the sweep method to differential equations of a hyperbolic type with partial derivatives with a small parameter that describes the motion in the process of gas lift [9, 11]. It is shown that when applying the sweep method to equations of motion, the coefficients of these functions are found using two differential equations. One of them is a quasilinear partial differential equation, and the second is the ordinary differential equation. The solution of the quasilinear equation is being found using the grid method [9,11]. Note that when constructing a solution to a quasilinear equation, it is impossible to find the coefficients of the positive degrees of the parameter ε . For this reason, the solution is found using coefficients of negative degrees of ε . The result is illustrated using an example taken from a specific practical problem [1, 9, 11].

Problem statement

It is known that [7,10,12] in the gas lift process, the system of differential equations of hyperbolic type with partial derivatives describing the movement of gas in the annular space and gas-liquid mixture in the lift has the following form

$$\begin{cases} \frac{\partial P_i(x,t)}{\partial t} = -\frac{c_i^2}{F_i} \cdot \frac{\partial Q_i(x,t)}{\partial x}, \\ \frac{\partial Q_i(x,t)}{\partial t} = -F_i \frac{\partial P_i(x,t)}{\partial x} - 2a_i Q_i(x,t) \end{cases} \quad i = 1, 2; t > 0, x \in (0, 2l) \quad (1)$$

Here $P_i(x,t)$ is the gas pressure in the well (in the lift is of the gas-liquid mixture), $Q_i(x,t)$ is the gas volume, c_i – speed of light, l – well depth, the parameter a_i is found using the expression $2a_i = \frac{g}{\omega_c} + \frac{\lambda \omega}{2D}$. In this expression λ is the coefficient of hydraulic resistance, g – acceleration of gravity, D – effective diameter of the annular space and the lift. Indexes 1 and 2 are parameters describing the movement in the annular space and in the lift, respectively.

If we solve system (1) by applying the straight lines method to it and determine the volume and pressure of the gas-liquid mixture at each point, then the number of equations in the system of differential equations will be too large, and this will lead to serious errors in computer calculations. For this reason, we consider the solution of the problem by introducing the parameter ε according to the relaxation method. I.e. in the system (1) we make the change $\varepsilon = \frac{1}{2l}$ and $z = \frac{x}{2l} = \varepsilon x$, where ε is a small parameter equal to the inverse value of the depth of the well [8]. As a result, from system (1) we obtain the following system of equations:

$$\begin{cases} \frac{\partial P_i(z, t, \varepsilon)}{\partial t} = -\frac{c_i^2}{F_i} \cdot \frac{\partial Q_i(z, t, \varepsilon)}{\partial z} \varepsilon, \\ \frac{\partial Q_i(z, t, \varepsilon)}{\partial t} = -F_i \frac{\partial P_i(z, t, \varepsilon)}{\partial z} \varepsilon - 2a_i Q_i(z, t, \varepsilon). \end{cases} \quad (2)$$

The boundary conditions of the resulting system of equations (2) are defined as follows [1,11]:

$$\begin{cases} P(0, t, \varepsilon) = P_0(t, \varepsilon), \\ Q(0, t, \varepsilon) = Q_0(t, \varepsilon), \end{cases} \quad (3)$$

By changing the role of the arguments z and t in this system, we can write equation (2) in the following form:

$$\begin{cases} \frac{\partial P(z, t, \varepsilon)}{\partial z} = -\frac{c^2}{F} \cdot \frac{\partial Q(z, t, \varepsilon)}{\partial t} \varepsilon, \\ \frac{\partial Q(z, t, \varepsilon)}{\partial z} = -F \frac{\partial P(z, t, \varepsilon)}{\partial t} \varepsilon - 2aQ(z, t, \varepsilon). \end{cases} \quad (4)$$

$z \in (-\infty; +\infty), t > 0.$

The initial conditions will take the form

$$\begin{cases} P(z, 0, \varepsilon) = P_0(z, \varepsilon), \\ Q(z, 0, \varepsilon) = Q_0(z, \varepsilon), \quad z \in (-\infty; +\infty). \end{cases} \quad (5)$$

We get the problem (4) and (5), which is equivalent to the problem (2) and (3). Using linearity (4), we can find the dependence of pressure $P(z, t, \varepsilon)$ on the volume of the gas-liquid mixture $Q(z, t, \varepsilon)$ in the following form [1,11]:

$$P(z, t, \varepsilon) = S(z, t, \varepsilon) \cdot Q(z, t, \varepsilon) + \alpha(t) \cdot R(z). \quad (6)$$

Here $\alpha(t)$ is an arbitrary scalar function satisfying the condition

$$\alpha(0) = 0, \int_0^\infty \alpha(t) dt = 1 \quad (7)$$

and $S(z, t, \varepsilon)$ and $R(z, t, \varepsilon)$ must be determined.

In the particular case $\alpha(t)$ can be selected as

$$\alpha(t) = te^{-t}.$$

To apply the sweep method, we obtain the derivative of (6) with respect to z and t .

$$\begin{cases} \frac{\partial P(z, t, \varepsilon)}{\partial z} = \frac{\partial S(z, t, \varepsilon)}{\partial z} Q(z, t, \varepsilon) + \\ + S(z, t, \varepsilon) \frac{\partial Q(z, t, \varepsilon)}{\partial z} + \alpha(t) R'(z), \\ \frac{\partial P(z, t, \varepsilon)}{\partial t} = \frac{\partial S(z, t, \varepsilon)}{\partial t} Q(z, t, \varepsilon) + \\ + S(z, t, \varepsilon) \frac{\partial Q(z, t, \varepsilon)}{\partial t} + \alpha'(t) R(z). \end{cases} \quad (8)$$

If we take into account the derivatives (8) in (4), we obtain

$$\begin{cases} \frac{\partial S(z, t, \varepsilon)}{\partial z} Q(z, t, \varepsilon) + S(z, t, \varepsilon) \frac{\partial Q(z, t, \varepsilon)}{\partial z} + \\ + \alpha(t) R'(z) = -\frac{c^2}{F} \frac{\partial Q(z, t, \varepsilon)}{\partial t} \varepsilon \\ \frac{\partial Q(z, t, \varepsilon)}{\partial z} = -F \varepsilon \left(\frac{\partial S(z, t, \varepsilon)}{\partial t} Q(z, t, \varepsilon) + \right. \\ \left. + S(z, t, \varepsilon) \frac{\partial Q(z, t, \varepsilon)}{\partial t} + \alpha'(t) R(z) \right) - 2aQ(z, t, \varepsilon). \end{cases} \quad (9)$$

If we substitute the derivative $\frac{\partial Q(z, t, \varepsilon)}{\partial z}$ from

the second equation of this system into the first equation, we obtain the following expression:

$$\begin{aligned} & \frac{\partial S(z, t, \varepsilon)}{\partial z} Q(z, t, \varepsilon) - \varepsilon FS(z, t, \varepsilon) \left(\frac{\partial S(z, t, \varepsilon)}{\partial t} Q(z, t, \varepsilon) + \right. \\ & \left. + S(z, t, \varepsilon) \frac{\partial Q(z, t, \varepsilon)}{\partial t} + \alpha'(t) R(z) \right) - \\ & - 2a S(z, t, \varepsilon) Q(z, t, \varepsilon) + \alpha(t) R'(z) = -\frac{c^2}{F} \frac{\partial Q(z, t, \varepsilon)}{\partial t} \varepsilon. \end{aligned} \quad (10)$$

If we simplify equation (10), then

$$\begin{aligned} & \left(\frac{\partial S(z, t, \varepsilon)}{\partial z} - \varepsilon FS(z, t, \varepsilon) \frac{\partial S(z, t, \varepsilon)}{\partial t} - \right. \\ & - 2a S(z, t, \varepsilon) Q(z, t, \varepsilon) + \left(\frac{c^2}{F} \varepsilon - \varepsilon FS^2 \right) \frac{\partial Q(z, t, \varepsilon)}{\partial t} - \\ & - \varepsilon FS(z, t, \varepsilon) \alpha'(t) R(z) + \alpha(t) R'(z) = 0. \end{aligned} \quad (11)$$

We integrate the obtained expression (11):

$$\begin{aligned} & \int_0^\infty \left[\left(\frac{\partial S(z, t, \varepsilon)}{\partial z} - \varepsilon FS(z, t, \varepsilon) \frac{\partial S(z, t, \varepsilon)}{\partial t} - \right. \right. \\ & - 2a S(z, t, \varepsilon) Q(z, t, \varepsilon) + \left(\frac{c^2}{F} \varepsilon - \varepsilon FS^2 \right) \frac{\partial Q(z, t, \varepsilon)}{\partial t} - \\ & \left. \left. - \varepsilon FS(z, t, \varepsilon) \alpha'(t) R(z) + \alpha(t) R'(z) \right] dt = 0 \end{aligned} \quad (12)$$

If to take into account this expression in (12), we obtain:

$$\begin{aligned} & \int_0^\infty \left[\frac{\partial S(z, t, \varepsilon)}{\partial z} - \varepsilon FS(z, t, \varepsilon) \frac{\partial S(z, t, \varepsilon)}{\partial t} - 2a S(z, t, \varepsilon) + \right. \\ & + 2\varepsilon FS(z, t, \varepsilon) \frac{\partial S(z, t, \varepsilon)}{\partial t} \left. \right] Q(z, t, \varepsilon) dt + \\ & + R'(z) - \varepsilon FR(z) \int_0^\infty S(z, t, \varepsilon) \alpha'(t) dt = 0. \end{aligned} \quad (13)$$

If we assume that the resulting expression (13) is satisfied regardless of $Q(z, t, \varepsilon)$, then we obtain

$$\begin{aligned} & \frac{\partial S(z, t, \varepsilon)}{\partial z} + \varepsilon FS(z, t, \varepsilon) \frac{\partial S(z, t, \varepsilon)}{\partial t} - \\ & - 2a S(z, t, \varepsilon) = 0 \end{aligned} \quad (14)$$

$$R'(z) - \varepsilon \left(\int_0^\infty S(z, t, \varepsilon) \alpha'(t) dt \right) FR(z) = 0 \quad (15)$$

The initial conditions will be as follows

$$S(x, 0, \varepsilon) = r(\varepsilon) = \frac{P_0(z, 0, \varepsilon)}{Q_0(z, 0, \varepsilon)}. \quad (16)$$

Consider the grid with step $h = \frac{L}{N}$ in the

direction of the axis z and with step $\tau = \frac{T}{M}$ in

the direction of the axis t of the numerical algorithm of quasilinear equation (14) with initial conditions (16), i.e.

$$z_i = ih, \quad i = \overline{0, N}, \quad hN = L,$$

$$t_j = j\tau, \quad j = \overline{0, M}, \quad \tau M = T.$$

Then, if we accept the notation

$$S(z_i, t_j) = S(ih, j\tau) = S_{ij},$$

we can write derivatives $\frac{\partial S(z, t, \varepsilon)}{\partial z}$ and $\frac{\partial S(z, t, \varepsilon)}{\partial t}$.

From the last for $S_{i,j+1}$ we get the expression

$$\begin{aligned} S_{i,j+1} &= S_{ij} + \frac{\tau}{\varepsilon Fh} (1 + 2ah) - \frac{\tau}{\varepsilon Fh} \frac{S_{i+1,j}}{S_{ij}} \\ i &= \overline{0, N-1}, \quad j = \overline{0, M-1}. \end{aligned} \quad (17)$$

After simple calculations (14) takes the following form

$$\tau S_{ij} - \tau S_{i-1,j} + \varepsilon Fh S_{ij} S_{i,j+1} - \varepsilon Fh S_{ij}^2 - 2ah \tau S_{ij} = 0$$

This makes it possible to determine $S_{i,j+1}$ from the following relation

$$\begin{aligned} S_{i,j+1} &= S_{ij} + (2ah - 1) \frac{\tau}{\varepsilon Fh} + \frac{\tau}{\varepsilon Fh} \frac{S_{i-1,j}}{S_{ij}} \\ i &= \overline{0, N-1}, \quad j = \overline{0, M-1}. \end{aligned} \quad (18)$$

Note that the approximate formulas (17) and (18) separately cannot be determined at all points of this grid. Therefore, we will try to combine the formulas (17) and (18) so that they can cover this grid as a whole

$$S(ih,0) = S_{i0} = \frac{P(ih)}{Q(ih)} = \frac{P_{i0}}{Q_{i0}}, \quad i = \overline{0, N}. \quad (19)$$

To cover all points of the grid, we will follow the following sequence:

1₁) from (17) for $i=0, j=0$ we will obtain

$$S_{01} = S_{00} + \frac{\tau}{\varepsilon Fh} (1+2ah) - \frac{\tau}{\varepsilon Fh} \frac{S_{10}}{S_{00}},$$

if to held a substitution

$$\tilde{S} = \frac{\tau}{Fh} (1+2ah) - \frac{\tau}{Fh} \frac{S_{10}}{S_{00}} \quad \text{here, we}$$

can write:

$$S_{01} = S_{00} + \tilde{S} \frac{1}{\varepsilon}. \quad (20)$$

1₂) from (18) for $i=1, j=0$ we will get

$$S_{11} = S_{10} + \frac{\tau}{\varepsilon Fh} (2ah-1) + \frac{\tau}{\varepsilon Fh} \frac{S_{00}}{S_{10}}, \quad (21)$$

We make the same replacement as in (20):

$$\tilde{S} = \frac{\tau}{Fh} (2ah-1) + \frac{\tau}{Fh} \frac{S_{00}}{S_{10}}.$$

Then expression (21) can be written as follows:

$$S_{11} = S_{10} + \tilde{S} \frac{1}{\varepsilon}. \quad (22)$$

Continuing this process, we determine S_{02} and S_{12} by the following method:

2₁) from (17) at $i=0, j=1$ we will get:

$$\begin{aligned} S_{02} &= S_{01} + \frac{\tau}{\varepsilon Fh} (1+2ah) - \frac{\tau}{\varepsilon Fh} \frac{S_{11}}{S_{01}} = \\ &= S_{00} + \frac{\tau}{\varepsilon Fh} (1+2ah) - \\ &\quad - \frac{\tau}{\varepsilon Fh} \frac{S_{10} + (2ah-1) \frac{\tau}{\varepsilon Fh} + \frac{\tau}{\varepsilon Fh} \frac{S_{00}}{S_{10}}}{S_{00} + \frac{\tau}{\varepsilon Fh} (1+2ah) - \frac{\tau}{\varepsilon Fh} \frac{S_{10}}{S_{00}}} \end{aligned} \quad (23)$$

By accepting $\frac{1}{\varepsilon} = \mu$ simplify the expression we will replace

$$\begin{aligned} W_1 &= \frac{\tau}{Fh} \frac{\mu S_{10} + \mu^2 (2ah-1) \frac{\tau}{Fh} + \mu^2 \frac{\tau}{Fh} \frac{S_{00}}{S_{10}}}{S_{00} + \mu \frac{\tau}{Fh} (1+2ah) - \mu \frac{\tau}{Fh} \frac{S_{10}}{S_{00}}} = \\ &= \frac{\tau}{Fh} \cdot \frac{S_{10}}{S_{00}} \cdot \frac{1}{\varepsilon} \end{aligned}$$

If we take into account this expression in (23), we obtain:

$$S_{02} = S_{01} + \tilde{S} \frac{1}{\varepsilon}.$$

2₂) at $i=1, j=1$, we will define S_{12} from (18) as follows

$$\begin{aligned} S_{12} &= S_{11} + \frac{\tau}{\varepsilon Fh} (2ah-1) + \frac{\tau}{\varepsilon Fh} \frac{S_{01}}{S_{11}} = \\ &= \frac{\tau}{\varepsilon Fh} \frac{S_{00} + (2ah+1) \frac{\tau}{\varepsilon Fh} - \frac{\tau}{\varepsilon Fh} \frac{S_{10}}{S_{00}}}{S_{10} + \frac{\tau}{\varepsilon Fh} (2ah-1) + \frac{\tau}{\varepsilon Fh} \frac{S_{00}}{S_{10}}}. \end{aligned} \quad (24)$$

Continuing this process along the vertical line at $j=\overline{1, M}$, we define all $S_{0,j}$ and $S_{1,j}$. Taking into account that at $j=\overline{1, M-1}$ all $S_{0,j}$ and $S_{1,j}$ are determined, at $i=0, j=M-1$ we get from (17)

$$S_{0M} = S_{0M-1} + \frac{\tau}{\varepsilon Fh} (1 + 2ah) - \frac{\tau}{\varepsilon Fh} \frac{S_{1M-1}}{S_{0M-1}}, \quad (25)$$

Using this, after expansion in series, we can write the last result in the form:

$$S_{0M} = S_{0M-1} + \tilde{S} \frac{1}{\varepsilon}.$$

Here $S_{0,M-1}$ is already known.

M_2) from (18) in $i=1, j=M-1$ we get

$$S_{1M} = S_{1M-1} + \frac{\tau}{Fh} (2ah - 1) + \frac{\tau}{Fh} \frac{S_{0M-1}}{S_{1M-1}}. \quad (26)$$

So continuing this process, we get:

$$S_{1M} = S_{1M-1} + \tilde{S} \frac{1}{\varepsilon}.$$

Thus, in $j = \overline{1, M}$ from the formulas (21) - (26) with initial conditions (16), we can determine all $S_{0,j}$ and $S_{1,j}$.

3_2) from (18) in $i=2, j=1$ we can determine S_{22} as follows

$$\begin{aligned} S_{22} &= S_{21} + \frac{\tau}{\varepsilon Fh} (2ah - 1) + \frac{\tau}{\varepsilon Fh} \frac{S_{11}}{S_{21}} = S_{21} + \\ &+ \frac{\tau}{\varepsilon Fh} (2ah - 1) + \frac{\tau}{\varepsilon Fh} \frac{S_{10} + (2ah - 1) \frac{\tau}{\varepsilon Fh} + \frac{\tau}{\varepsilon Fh} \frac{S_{00}}{S_{10}}}{S_{10} + \frac{\tau}{\varepsilon Fh} (2ah - 1) + \frac{\tau}{\varepsilon Fh} \frac{S_{10}}{S_{20}}}. \\ S_{22} &= S_{21} + \frac{\tau}{\varepsilon Fh} (2ah - 1) + \frac{\tau}{\varepsilon Fh} \frac{S_{10}}{S_{20}}. \end{aligned}$$

Then, starting from (18) in $i=2, j=\overline{0, M-1}$ we will obtain all $S_{2,j}$. Continuing this process, in $j=\overline{1, M}$ we find all $S_{i,j}$

$$S_{ij} = S_{ij-1} + \tilde{S} \frac{1}{\varepsilon}, \quad i = \overline{2, N}, \quad j = \overline{1, M}. \quad (27)$$

As can be seen from the above scheme, two types of patterns were used to calculate

$S_{i,j}$ at all nodal points. One of them discretizes the derivative the step z forward, the other one step z back.

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**RELAKSASIYA ÜSULUNDAN İSTİFADƏ EDƏRƏK KVAZİXƏTTİ
TƏNLİYİN HƏLLİ ÜÇÜN HESABLAMA ALQORİTMİ****R.M.Tağıyev, O.Z.Namazov**

Bu işdə neftin qaz-lift üsulu ilə hasilatının modeləşdirilməsində qarşıya çıxan böyükparametrlı xüsusi törəməli diferensial tənliklər sistemi üçün Koşi məsələsinə baxılır. Daha sonra alınmış həllərin hesablanması üçün relaksasiya üsulundan istifadə edilir. Bu formada məsələnin araşdırılması həll üsulunu xeyli sadələşdirir və imkan verir ki, təklif olunan həll alqoritmlərinin kompüter realizasiyasını sona kimi apara bilək.

Açar sözlər: qaz-lift, hiperbolik tənliklər, diferensial tənliklər, qovma üsulu, şəbəkə üsulu, relaksasiya üsulu

**ВЫЧИСЛИТЕЛЬНЫЙ АЛГОРИТМ ДЛЯ РЕШЕНИЯ КВАЗИЛИНЕЙНОГО УРАВНЕНИЯ
С ПОМОЩЬЮ МЕТОДА РЕЛАКСАЦИИ****Р.М.Тагиев, О.З.Намазов**

В работе рассматривается задача Коши для системы дифференциальных уравнений с частными производными с большим параметром, описывающих моделирование добычи нефти методом газлифта. Затем для вычисления полученных решений используется метод релаксации. Исследование задачи в такой форме упрощает метод решения и дает возможность довести до конца компьютерную реализацию предлагаемых алгоритмов вычисления.

Ключевые слова: газлифт, гиперболические уравнения, дифференциальные уравнения, метод прогонки, метод сеток, метод релаксации