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ON A NEW METHOD OF INVESTIGATION OF EXTREMAL MANIFOLDS

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In the present paper we offer a new approach for proving of the extremality of some manifolds. Basic results in this direction was first proved by V. G. Sprindzuk by his method of essential and nonessential domains. We show that it is possible to derive the result of Sprindzuk from the theorems on convergence exponent for improper integrals like the special integral of Tarry's problem.

Keywords: Diophantine approximation, algebraic variety, extremal variety, convergence exponent

Introduction

Let's introduce into consideration following set of polynomials with integral coefficients of degree not exceeding n :

$$\Pi = \left\{ f(x) = \sum_{i=1}^n a_i x^i \mid a_i \in \mathbf{R} \right\}.$$

We call the number

$$h(f) = \max(|a_0|, |a_1|, \dots, |a_n|)$$

to be the height of the polynomial $f(x)$. Let we are given with real transcendental number α (consequently α cannot be a root of any polynomial from Π). Let $h > 0$ be a real number. Mahler K. showed that there is a constant $\kappa > 0$ for which the inequality

$$|f(\alpha)| > h^{-n\gamma}; \quad h = h(f) \quad (1)$$

is satisfied for any polynomial $f(x)$ with the height $\leq h$ for almost all real numbers. He had found the value $\kappa = 4 + \varepsilon$ ($\varepsilon > 0$ is any). Mahler K. conjectured that it is possible to take $\kappa = 1 + \varepsilon$.

In the work (see [3,7]) Khintschine A. showed that the system of inequalities

$$\max(\|tq\|, \|t^2q\|, \dots, \|t^nq\|) < \delta q^{-1/n}$$

has infinite and fixed set M of solutions in positive integral numbers $q > 0$ for almost all real t in the Lebesgue sense. It means that for almost all real t there are infinitely many natural numbers $q_1, q_2, \dots, q_m, \dots$ such that for every natural m the following inequalities are satisfied:

$$\left| t^k - \frac{a_k}{q_m} \right| \leq \delta q_m^{-1/n}, \quad k = 1, \dots, n.$$

To define basic notions of the theory, we consider the system of inequalities

$$\max(\|\alpha_1 q\|, \|\alpha_2 q\|, \dots, \|\alpha_n q\|) < q^{-u}, \quad u > 0. \quad (1)$$

Let $u(\alpha_1, \dots, \alpha_n)$ be defined as a *sup* of such $u > 0$ for which (1) is satisfied for infinite set of natural numbers q . It is not difficult to show that $u(\alpha_1, \dots, \alpha_n) \geq 1/n$ (see [10]). From this definition it follows that the inequality (1) is satisfied for infinitely many natural numbers q when $u < 1/n$. When $u(\alpha_1, \dots, \alpha_n) = 1/n$ for almost all points of the variety $(\alpha_1, \dots, \alpha_n) \in R^n$ of less dimension, then we call this manifold as an *extremal manifold*.

In 1993 Karatsuba A. A. advanced an opinion that the question on extremality of

some algebraic varieties could be investigated by using of results on convergence exponent in the Tarry's problem (concerning the problem itself see [1, 4-5]). In the present work we show that this proposition is valid and can be useful for cases of many other manifolds. We prove some results of Sprindzuk V. G. ([10-11]) by a new method.

Auxiliary results

We shall use following two auxiliary lemmas: Borel-Kantelly's lemma 1 and Kovalevskaya's lemma 2 (see[2,6,10]).

Lemma 1. Let A_q ($q = 1, 2, \dots$) be a sequence of measurable sets in $\mathbf{R}^n$, and

$$\sum_{q=1}^{\infty} \text{mes} A_q < \infty.$$

Then the measure of such real numbers which fall into infinite number of sets A_q is zero.

Lemma 2. Let q be natural numbers, $f_j(\bar{x}), j = 1, \dots, N$ be a family of real measurable functions defined in the cube $\Omega = [0, 1]^r$. Denote by $\mu(q)$ the measure of a set of that $\bar{x} \in \Omega = [0, 1]^r$ for which

$$\|f_j(\bar{x})\| < q^{-r_j} \quad (1 \leq j \leq N).$$

Then,

$$\mu(q) < q^{-r} \sum_{|c_1| < q^{r_1}} \dots \sum_{|c_N| < q^{r_N}} \left| \int_{\Omega} e^{2\pi i(c_1 f_1(\bar{x}) + \dots + c_N f_N(\bar{x}))} d\bar{x} \right|;$$

here $r = r_1 + \dots + r_N$, and the constant in the symbol $<$ depends on N only.

Basic results

In this section we formulate some results on extremality of manifolds which we prove by a new method.

Theorem 1. Let $n_1, n_2, \dots, n_k$ be a natural number, $P(x_1, x_2, \dots, x_k)$ be a polynomial with integral coefficients of degrees n_i with respects variables x_i ($i = 1, 2, \dots, k$), i. e.

$$P(x_1, x_2, \dots, x_k) = \sum_{\substack{0 \leq i_1 \leq n_1 \\ \dots \\ 0 \leq i_k \leq n_k}} a_{i_1 i_2 \dots i_k} x_1^{i_1} x_2^{i_2} \dots x_k^{i_k}.$$

Let further $v(\beta_1, \beta_2, \dots, \beta_k)$ means for the real vector $\bar{\beta} = (\beta_1, \beta_2, \dots, \beta_k)$ the supremum of such $v > 0$, for which the inequality

$$|P(\beta_1, \beta_2, \dots, \beta_k)| < h^{-v}$$

is satisfied for infinitely many polynomials P , where

$$h = \max_{i_1 i_2 \dots i_k} \left| a_{i_1 i_2 \dots i_k} \right| \begin{pmatrix} 0 \leq i_1 \leq n_1 \\ \dots \\ 0 \leq i_k \leq n_k \end{pmatrix}.$$

Then for almost all $\bar{\beta} = (\beta_1, \beta_2, \dots, \beta_k)$

$$v(\beta_1, \beta_2, \dots, \beta_k) = (n_1 + 1)(n_2 + 1) \dots (n_k + 1) - 1.$$

Theorem 2. Let $m_1 < m_2 < \dots < m_n$ be natural numbers, ω be a transscendental number, $v_n(\omega) = v(m_1, m_2, \dots, m_n)$ be a supremum of such $v > 0$ for which the inequality $|a_0 + a_1 \omega^{m_1} + \dots + a_n \omega^{m_n}| < h^{-v}$, $h = \max(|a_0|, |a_1|, \dots, |a_n|)$

has infinitely many solutions in integral numbers $a_0, a_2, \dots, a_n$. Then, $v_n(\omega)$.

Let we are given with some continuously differentiable r -dimensional manifold $\Gamma = (f_1(\bar{x}), \dots, f_n(\bar{x})), \bar{x} \in \Omega = [0, 1]^r, r < n$. Taking natural number such that $rh > n$ consider the map

$$\varphi_j : \Omega^h \rightarrow \mathbf{R}^n$$

defined by the equalities

$$\varphi_j(\bar{x}) = \varphi_j(\bar{x}_1, \dots, \bar{x}_h) = f_j(\bar{x}_1) + \dots + f_j(\bar{x}_h); \bar{x}_s = (x_{s1}, \dots, x_{sr})$$

Let the Jacoby matrix of the map $(\bar{x}_1, \dots, \bar{x}_h) \mapsto (\varphi_1(\bar{x}), \dots, \varphi_h(\bar{x}_h))$ i. e. the matrix composed of the gradients of the functions $\varphi_1(\bar{x}), \dots, \varphi_h(\bar{x}_h)$ be the matrix of maximal rank. It is easy to see that the Jacoby matrix has a view

$$\begin{pmatrix} \frac{\partial \varphi_1}{\partial x_{11}} & \dots & \frac{\partial \varphi_1}{\partial x_{1r}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_n}{\partial x_{11}} & \dots & \frac{\partial \varphi_n}{\partial x_{1r}} \end{pmatrix}.$$

Theorem 3. If the integral

$$\int_{-\infty-\infty}^{\infty} \dots \int_{-\infty-\infty}^{\infty} \left| \int_0^1 \dots \int_0^1 e^{2\pi i(\alpha_n f_n(\bar{x}) + \alpha_{n-1} f_{n-1}(\bar{x}) + \dots + \alpha_1 f_1(\bar{x}))} d\bar{x} \right|^{2h} d\alpha_n d\alpha_{n-1} \dots d\alpha_1$$

is convergent for some integral number $h > 0$ then the manifold Γ is extremal.

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REFERENCES

1. Arkhipov G. I., Karatsuba A. A., Chubarikov V. N. Theory of multiple trigonometric Sums., M.: Nauka, 1987.
2. Bernik V. I., Kovaleuskaya E. I. Extremality property of some surfaces in n dimensional Euclidean space., Matemat. Zametki, (1974), v.15, no 2, 247-254.

3. Cassels J. W. S. An introduction to Diophantine Approximation. IIL, Moscow, 1961.
4. Jabbarov I. Sh. The exponent of convergence of a special integral in multidimensional problems Terry, Chebyshevskii Sbornik, 14:2 (2013), 74-103.
5. Jabbarov I. Sh. On an identity of Harmonic Analysis and its applications, Dokl.AS USSR, (1990), v.314, no 5, 1052-1054.
6. Kovaleuskaya E.I. Simultaneously extremal manifolds., Matemat. Zametki, (1987),v.41, no 1, 3-8.
7. Khintschine A. Über eine Klasse linearer Diophantischer Approximationen, Circolo mat. Palermo 50, 1926, 175-195.
8. Mahler K. Zur Approximation der Exponentialfunktionen und des Logarithmus. I, II., J. reine und angew. Math., 1932, 166, S. 118150.
9. Mahler K. Über das Mass der Menge aller S-Zahlen. Math. Ann., 1932, v. 106 pp.131 139.
10. Sprindzuk V. G. Metric theory of Diophantine Approximations., M: Nauka, 1977.
11. Sprindzuk V. G. A proof of Mahler's conjecture on the measure of the set of S-numbers. Izvestiya: Mathematics, 16: 1 (1981), 2140.

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EKSTREMAL ÇOXOBRAZLILARIN YENİ ÜSULLA TƏDQİQİ HAQQINDA

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Baxılan işdə biz bəzi çoxobrazlıların ekstremallıq məsələsinin yeni üsulla tədqiqini veririk. Bu istiqamətdə əsas nəticələr V. Q.Sprincuk tərəfindən onun mühüm və qeyri-mühüm oblastlar üsulu ilə verilmişdir. Biz göstəririk ki, bu məsələlər Terri probleminə məxsusi inteqrala oxşar qeyri-məxsusi inteqralların yığılma göstəricisi haqqında teoremin tətbiqi ilə də həll oluna bilər.

Açar sözlər: Diofant yaxınlaşmaları, cəbri çoxobrazlı, extremal çoxobrazlı, yığılma göstəricisi.

О НОВОМ МЕТОДЕ ИССЛЕДОВАНИЯ ЭКСТРЕМАЛЬНЫХ МНОГООБРАЗИЙ

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В настоящей статье рассматривается новый подход для решения вопроса экстремальности многообразий. Основные результаты в этом направлении впервые были доказаны Спринджуком В. Г. его методом существенных и несущественных областей. Мы показываем, что эти вопросы могут быть изучены с помощью теорем о показателе сходимости несобственных интегралов, подобных особому интегралу проблемы Терри.

Ключевые слова: Диофантовы приближения, алгебраические многообразия, экстремальные многообразия, показатель сходимости.