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SOLUTION OF ONE MIXED PROBLEM FOR A NONHOMOGENEOUS HEAT EQUATION WITH TIME ADVANCE IN THE BOUNDARY CONDITIONS

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In this paper one-dimensional mixed problem for a nonhomogeneous heat equation with time advance in nonlocal boundary conditions is studied. Under minimal conditions on the data, the unique solvability is proved and an explicit analytic representation is obtained for the solution.

Keywords: mixed problem, time advance, residue method

Statement of the problem

Suppose, that $L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)u(x,t) = u_t(x,t) - a^2 u_{xx}(x,t)$,

$l_j u(x,t) \equiv u(x,t + (1-j)\omega) + \alpha_j u(1-x, t + j\omega)$
($j = 0, 1$) and

$l_j u(x,t) \equiv a_{j-2} u_x^{(j-2)}(x,t) + b_{j-2} u_x^{(j-2)}(1-x, t)$
($j = 2, 3$), where $a, \omega, \alpha_j, a_j, b_j$ ($j = 0, 1$) are real-valued constants, $a \neq 0, \omega > 0, \alpha_0 \cdot \alpha_1 \neq 0$.

In semi-stripe $\Pi = \{(x,t); 0 < x < 1, t > 0\}$ the following mixed problem is considered

$$Lu(x,t) = f(x,t), (x,t) \in \Pi, \quad (1)$$

$$u(x,0) = \varphi(x), 0 < x < 1, \quad (2)$$

$$\begin{aligned} l_j u(x,t) \Big|_{x=0} &= \psi_j(t), \\ t > 0, j &= 0, 1, \end{aligned} \quad (3)$$

$$l_j u(x,t) \Big|_{x=\omega} = \psi_j(t), 0 \leq t \leq \omega, j = 2, 3, \quad (4)$$

where $f(x,t), \varphi(x), \psi_k(t)$ ($k = 0, 1, 2, 3$) are given and $u(x,t)$ desired are functions.

As a solution of the problem (1)-(4) we understand function $u(x,t)$, which is

- 1) $u(x,t) \in C^{2,1}(\Pi) \cap C(0 < x < 1, t \geq 0)$;
- 2) $l_j u(x,t) \in C(0 \leq x < 1, t > 0), (j = 0, 1)$;
- 3) $l_j u(x,t) \in C(0 \leq x < 1, 0 \leq t \leq \omega),$
($j = 3, 4$);

4) $u(x,t)$ satisfies equalities (1)-(4) in common sense.

Among the large number of works devoted to general and qualitative problems for equations with deviating arguments, as well as works for the heat equation, where the functional conditions are posed instead of the boundary conditions, we note [5-11]. The mixed problem for the heat equation with a deviation of the argument in the boundary conditions was considered in [12]. The problem (1) - (4) is not covered by these studies.

Here, by combining the residue method and the method of the contour integral of academician M.L. Rasulov [2], [3], prove the existence and uniqueness of the solution of the mixed problem (1) - (4) and obtain an explicit analytic representation for it.

The uniqueness of the solution. Consider the mixed problem (1), (2), (4) in the rectangle $0 < x < 1, 0 < t < \omega$.

Using substitution

$$u(x,t) = v(x,t) + w(x,t) \quad (5)$$

from this problem we pass to the next mixed problem under homogeneous boundary conditions

$$Lw(x,t) = F(x,t), 0 < x < 1, 0 < t < \omega, \quad (6)$$

$$w(x,0) = f(x), 0 < x < 1 \quad (7)$$

$$l_j w(x, t) \Big|_{x=0} = 0, 0 \leq t \leq \omega, \quad j = 2, 3, \quad (8)$$

where

$$F(x, t) = f(x, t) - \frac{\psi_2'(t)}{a_0 + b_0} + \frac{b_0 \psi_3'(t)}{(a_0 + b_0)(a_1 + b_1)} - \frac{x \psi_3'(t)}{a_1 + b_1}, \quad (9)$$

$$f(x) = \varphi(x) - \frac{\psi_2(0)}{a_0 + b_0} + \frac{b_0 \psi_3(0)}{(a_0 + b_0)(a_1 + b_1)} - \frac{x \psi_3(0)}{a_1 + b_1}, \quad (10)$$

$$v(x, t) = \frac{\psi_2(t)}{a_0 + b_0} - \frac{b_0 \psi_3(t)}{(a_0 + b_0)(a_1 + b_1)} + \frac{x \psi_3(t)}{a_1 + b_1}. \quad (11)$$

According to the general scheme of the residue method, first the mixed problem (6) - (8) is matched by two problems with a complex parameter μ .

A. The Cauchy problem

$$\frac{dT}{dt} - \mu^2 T = F(x, t), \quad 0 \leq t \leq \omega, \quad (12)$$

$$T(x, 0) = f(x), \quad 0 < x < 1. \quad (13)$$

B. The spectral problem

$$a^2 y'' - \mu^2 y = h(x), \quad (14)$$

$$l_j y(x, \mu) \Big|_{x=0} = 0, \quad (15)$$

where $h(x)$ - some continuous function on $[0, 1]$.

If $a_0 b_1 + b_0 a_1 \neq 0$ then for all complex value $\mu \neq \mu_\nu$, where $\mu_\nu = a \ln_0 \left(B \pm \sqrt{B^2 - A^2} \right) A^{-1} + 2a\pi\nu i$, $\nu = 0, \pm 1, \pm 2, \dots$, $A = a_0 b_1 + b_0 a_1$, $B = -a_0 a_1 - b_0 b_1$, there exists Green's function $G_1(x, \xi, \mu)$ of problem (14), (15), analytical on μ everywhere, except for points μ_ν that are its poles. It is clear that $|\mu_\nu| \rightarrow \infty$ ($\nu \rightarrow \infty$), there exist $h > 0$, $\delta > 0$ such that

$$-h < \operatorname{Re} \mu_\nu < h, \quad |\mu_{\nu+1} - \mu_\nu| > 2\delta, \quad (\nu = 1, 2, \dots). \quad (16)$$

Outside $\delta > 0$ - neighborhoods of points μ_ν , the estimates are valid

$$\left| \frac{\partial^k G_1(x, \xi, \mu)}{\partial x^k} \right| \leq C_0 |\mu|^{k-1}, \quad C_0 > 0, \quad k = 0, 1, 2. \quad (17)$$

Take the following notations:

Suppose, that $b > 0$, $r > 0$, $z = x + iy$ complex variable, $L_b = \{z : \operatorname{Re} z^2 = b\}$ - hyperbola with branches $L_b^\pm = \{z : \operatorname{Re} z^2 = b, \pm \operatorname{Re} z > 0\}$, $\Omega_r = \{z : |z| = r\}$.

Introduce also contours (broken line)

$$\hat{L}_b = \hat{L}_b^+ \cup \hat{L}_b^-,$$

$$\hat{L}_b^\pm = \left\{ z : \pm z = \sigma e^{\frac{3\pi i}{8}}, \sigma \in \left[2b\sqrt{1+\sqrt{2}}, \infty \right) \right\} \cup \left\{ z : \pm z = c(1+i\eta), \eta \in [-1-\sqrt{2}, 1+\sqrt{2}] \right\} \cup \left\{ z : \pm z = \sigma e^{\frac{3\pi i}{8}}, \sigma \in \left[2b\sqrt{1+\sqrt{2}}, \infty \right) \right\}.$$

Let $\{r_\nu\}$ be a sequence of numbers such that

$$0 < r_1 < r_2 < \dots < r_\nu < \dots, \quad \lim_{\nu \rightarrow \infty} r_\nu = +\infty$$

and circles Ω_{r_ν} do not cross $\delta > 0$ - the neighborhood μ_ν .

Using the scheme of the residue method [3] for an ordinary problem (a problem without time advance) (6) - (8) it is proved that if it has a solution, then this solution is represented by the formula

$$w(x, t) = \sum_{\nu=1}^{\infty} \operatorname{res}_{\mu_\nu} \mu \int_0^1 G_1(x, \xi, \mu) T(\xi, \mu, t) d\xi, \quad (18)$$

and by transformation and justification of this formula, taking into account (5) for problem (1), (2), (4) (in the rectangle $0 < x < 1$, $0 < t \leq \omega$) is proved

Theorem 1. Let $a_0 b_1 + b_0 a_1 \neq 0$, $\varphi(x) \in C^2[0, 1]$, $l_j \varphi \Big|_{x=0} = 0$ ($j = 2, 3$),

$f(x, t) \in C^{2,0}([0, 1] \times [0, \omega])$, $l_j f|_{x=0} = 0$,
 $(j = 2, 3)$, $\psi_k(t) \in C^1[0, \omega]$, $(k = 2, 3)$ and
 $(a_0 + b_0)(a_1 + b_1) \neq 0$. Then the problem (1),
 (2), (4) has a unique solution and it is represented by the formula

$$\begin{aligned} u(x, t) = & \frac{\psi_2(t)}{a_0 + b_0} - \frac{b_0 \psi_3(t)}{(a_0 + b_0)(a_1 + b_1)} + \frac{x \psi_3(t)}{a_1 + b_1} + \\ & + \varphi(x) - \frac{\psi_2(0)}{a_0 + b_0} + \frac{b \psi_3(0)}{(a_0 + b_0)(a_1 + b_1)} - \frac{x \psi_3(0)}{a_1 + b_1} + \\ & + \frac{a^2}{\pi i} \int_{\tilde{L}_h^+} \mu^{-1} e^{\mu^2 t} d\mu \int_0^1 G_1(x, \xi, \mu) \left[\varphi''(\xi) + \int_0^t e^{-\mu^2 \tau} f_{\xi\xi}''(\xi, \tau) d\tau \right] d\xi + \\ & + \int_0^t \left[f(x, \tau) - \frac{\psi_2'(\tau)}{a_0 + b_0} + \frac{b_0 \psi_3'(\tau)}{(a_0 + b_0)(a_1 + b_1)} - \frac{x \psi_3'(\tau)}{a_1 + b_1} \right] d\tau \end{aligned} \quad (19)$$

at $0 \leq x \leq 1$, $0 \leq t \leq \omega$, where h a number from (16). From this theorem the following is true:

Theorem 2. Under the conditions of Theorem 1, problem (1) - (4) cannot have more than one solution.

Indeed, if problem (1) - (4) would have two solutions $u_1(x, t)$, $u_2(x, t)$, then their difference $v(x, t) = u_1(x, t) - u_2(x, t)$ would be the solution of the homogeneous problem (with $\varphi(x) \equiv 0$, $f(x, t) \equiv 0$, $\psi_k(t) = 0$, $k = \overline{1, 4}$) (1)-(4), and by that the homogeneous problem (1), (2), (4) in $\{(x, t) : 0 \leq x \leq 1, 0 \leq t \leq \omega\}$. Then, by virtue of (19), $v(x, t) \equiv 0$ at $0 \leq x \leq 1$, $0 \leq t \leq \omega$ and from conditions (3) it follows that $v(0, t) = v(1, t) \equiv 0$, at $t \geq 0$, whence, taking into account the maximum principle [1], we conclude that $v(x, t) \equiv 0$ в $\{(x, t) : 0 \leq x \leq 1, t \geq 0\}$.

The existence and presentation of the solution. Now consider another spectral problem:

$$\begin{aligned} L\left(\frac{d}{dx}, \lambda^2\right) z(x, \lambda) &= 0 \\ z(0, \lambda) &= 0, \quad z(1, \lambda) = 0 \end{aligned} \quad (20)$$

with complex parameter λ . It is known [4] that it has the Green function $G_2(x, \xi, \lambda)$ which is analytic by λ everywhere, except the points

$\lambda_k = a\pi k i$ ($k = \pm 1, \pm 2, \dots$) which are its simple poles.

Introduce the function

$$\begin{aligned} Q(x, \lambda, p(\lambda), q(\lambda)) = & \left(e^{-\frac{\lambda}{a}} - e^{\frac{\lambda}{a}} \right)^{-1} \left[e^{\frac{\lambda}{a} x} \left(p(\lambda) e^{-\frac{\lambda}{a}} - q(\lambda) \right) + \right. \\ & \left. + e^{-\frac{\lambda}{a} x} \left(q(\lambda) - p(\lambda) e^{\frac{\lambda}{a}} \right) \right], \end{aligned} \quad (21)$$

where

$$\begin{aligned} p(\lambda) = & \left(\alpha_1 e^{2\lambda^2 \omega} - \alpha_0 \right)^{-1} \left[\alpha_1 e^{\lambda^2 \omega} \left(\tilde{\psi}_0(\lambda) + e^{\lambda^2 \omega} \int_0^\omega e^{-\lambda^2 t} u(0, t) dt \right) - \right. \\ & \left. - \alpha_0 \left(\tilde{\psi}_1(\lambda) + \alpha_1 e^{\lambda^2 \omega} \int_0^\omega e^{-\lambda^2 t} u(1, t) dt \right) \right], \end{aligned} \quad (22)$$

$$\begin{aligned} q(\lambda) = & \left(\alpha_1 e^{2\lambda^2 \omega} - \alpha_0 \right)^{-1} \left[e^{\lambda^2 \omega} \left(\tilde{\psi}_1(\lambda) + \alpha_1 e^{\lambda^2 \omega} \int_0^\omega e^{-\lambda^2 t} u(1, t) dt \right) - \right. \\ & \left. - \left(\tilde{\psi}_0(\lambda) + e^{\lambda^2 \omega} \int_0^\omega e^{-\lambda^2 t} u(0, t) dt \right) \right], \end{aligned} \quad (23)$$

where

$$\tilde{\psi}_k(\lambda) = \int_0^\omega e^{-\lambda^2 t} \psi_k(t) dt, \quad (k = 0, 1), \quad (24)$$

$u(s, t)$ the boundary value of the solution on the parts $\{(s, t) : 0 \leq t \leq \omega, s = 0, 1\}$ of the lateral boundary of the domain $\{(x, t) : 0 < x < 1, t > 0\}$, which are determined from (19).

$$\text{Let's fix the number } C_1 > \max \left(0, \ln \left| \frac{\alpha_0}{\alpha_1} \right| \right).$$

It is proved

Theorem 3. Suppose, that $a_0 b_1 + b_0 a_1 \neq 0$,
 $\varphi(x) \in C^2[0, 1]$, $l_j \varphi|_{x=0} = 0$, $(j = 0, 1)$,
 $f(x, t) \in C^{2,0}([0, 1] \times [0, \infty))$, $l_j f|_{x=0} = 0$,

($j = 0, 1$), $\psi_j(t) \in C^1[0, \infty)$, ($j = 0, 1$) and $(a_0 + b_0)(a_1 + b_1) \neq 0$. Then the problem (1)-(4) has the classical solution and it is represented by the formula

$$u(x, t) = \varphi(x) - \int_0^t f(x, \tau) d\tau + \\ + \frac{a^2}{\pi i} \int_{L_{C_1}^+} \lambda^{-1} e^{\lambda^2 t} \left[\int_0^1 G_2(x, \xi, \lambda) \left(\varphi''(\xi) + \int_0^t e^{-\lambda^2 \tau} f_{\xi\xi}''(\xi, \tau) d\tau \right) d\xi - \right. \\ \left. - Q(x, \lambda, \varphi(0), \varphi(1)) d\lambda + \frac{1}{\pi i} \int_{L_{C_1}^+} \lambda e^{\lambda^2 t} Q(x, \lambda, p(\lambda), q(\lambda)) d\lambda \right]. \quad (26)$$

Remark. Combining Theorems 1 and 2, we obtain the final statement: Under the conditions of Theorems 1 and 2, the main problem (1) - (4) has a unique classical solution and it is represented by formula (26).

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SƏRHƏD ŞƏRTLƏRİNDƏ ZAMANA GÖRƏ GECİKMƏ OLAN HALDA QEYRİ-BİRCİNS İSTİLİKKEÇİRMƏ TƏNLİYİ ÜÇÜN QARIŞIQ MƏSƏLƏNİN HƏLLİ

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Bu işdə yarımzolaqda sərhəd şərtlərində zamana görə sürüşmə olan halda qeyri-bircins istilikkeçirmə tənliyi üçün birölçülü qarışıq məsələyə baxılmışdır. Məsələnin məlumatları üzərinə minimal şərtlər daxilində həllin varlığı və yeganəliyi isbat olunmuş və həll üçün aşkar analitik ifadə alınmışdır.

Açar sözlər: qarışıq məsələ, zamana görə gecikmə, çıxıqlar üsulu

РЕШЕНИЕ ОДНОЙ СМЕШАННОЙ ЗАДАЧИ ДЛЯ НЕОДНОРОДНОГО УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ СО СМЕЩЕНИЕМ ВРЕМЕНИ В ГРАНИЧНЫХ УСЛОВИЯХ

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В этой работе исследуется одномерная смешанная задача для неоднородного уравнения теплопроводности, с отклонением времени в нелокальных граничных условиях. При минимальных условиях на данные доказана однозначная разрешимость и получено явное аналитическое представление для решения.

Ключевые слова: смешанная задача, отклонение по времени, вычетный метод