

UDC 517.9

THE SUBDIFFERENTIAL OF THE FUNCTIONAL IN TWO-DIMENSIONAL ABSOLUTELY CONTINUOUS FUNCTIONS

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The subdifferential of a functional of a special form in the space of continuous functions, in the space of absolutely continuous and two-dimensional absolutely continuous functions in the work studies. The work studies the subdifferential of the integral and terminal functionals on the space of two-dimensional absolutely continuous functions. Two different norms are considered in the space of absolutely continuous and two-dimensional absolutely continuous functions, showing that the subdifferentials in these norms coincide. Using these equalities, subdifferentials of integral functionals are studied. Next, an analogue of the lemma of the classical calculus of variations for two-dimensional absolutely continuous functions is proved. Using this lemma, we study the subdifferential of the terminal functional in the class of two-dimensional absolutely continuous functions.

Keywords: subdifferential, integral functional, terminal functional, function, set

1. Introduction

The work studies the subdifferential of the integral and terminal functionals in space of two-dimensional absolutely continuous functions. The work is a generalization of some results of the work in [1,2]. Such a problem in $W_{p,1}^n[t_0, T]$ studied in [3]-[5].

When studying nonsmooth variational problems, an important role is played by the study of the conjugate function and the subdifferential of the integral functional. In [6-9], the subdifferential of a convex integral functional in spaces of two-dimensional absolutely continuous functions was studied. This paper studies the subdifferential of the integral functional in the class of two-dimensional absolutely continuous functions and its subspaces. The paper shows that the subdifferential of the integral functional in the class of two-dimensional absolutely continuous functions is closely related to the subdifferential of the integral functional in the class of continuous functions. The subdifferential of a terminal functional and the sum of several functionals from the class of two-dimensional absolutely continuous functions are also studied.

If $1 \leq p \leq \infty$, $d \in \mathbb{R}^n$, $x(\cdot) \in L_p^n[0, T]$, $y(\cdot) \in L_p^n[0, S]$ and $z(\cdot) \in L_p^n([0, T] \times [0, S])$, then a set of functions that have the form

$$u(t, s) = d + \int_0^t x(\tau) d\tau + \int_0^s y(v) dv + \int_0^t \int_0^s z(\tau, v) d\tau dv$$

denoted by $A_p^n([0, T] \times [0, S])$.

It follows that $u(0, 0) = d$, $u_t(t, 0) = x(t)$, $u_s(0, s) = y(s)$ and $u_{ts}(t, s) = z(t, s)$.

It follows that there exist $(c, a(\cdot), b(\cdot), v(\cdot)) \in \mathbb{R}^n \times L_p^n[0, T] \times L_p^n[0, S] \times L_p^n([0, T] \times [0, S])$, $\frac{1}{p} + \frac{1}{p'} = 1$, such that every linear continuous functional v^* on $A_p^n([0, T] \times [0, S])$, $1 \leq p < \infty$, (i.e. $v^* \in A_p^n([0, T] \times [0, S])^*$) has the form:

$$v^*(u) = (c|u(0, 0)) + \int_0^T (a(t)|u_t(t, 0)) dt + \int_0^S (b(s)|u_s(0, s)) ds + \int_0^T \int_0^S (v(t, s)|u_{ts}(t, s)) dt ds.$$

For simplicity, the functional v^* is denoted by $(c, a(\cdot), b(\cdot), v(\cdot))$, i.e. $v^* = (c, a(\cdot), b(\cdot), v(\cdot))$. The set $A_p^n([0, T] \times [0, S])$, $1 \leq p \leq \infty$, with norm

$$\|u\|_{C_p^n} = \max_{(t,s) \in [0,T] \times [0,S]} \|u(t, s)\| \quad \text{denote by}$$

$C_p^n([0, T] \times [0, S])$, where $\|u(t, s)\| = \left(\sum_{i=1}^n u_i(t, s)^2 \right)^{\frac{1}{2}}$.

2. About subdifferential of the integral functional in $C^n(D)$

Let be D a compact metric space. Through $C^n(D)$ denoted by Banach space continuous mappings (vector functions) from D in R^n with the norm $\|x(\cdot)\|_{C^n} = \max_{t \in D} \|x(t)\|$. Through

$\text{frm}(D)$ denoted by vector space of all Radon measures in D (see [3, p.65]). There is isomorphic correspondence between spaces $C^n(D)^*$ and $\text{frm}(D)^n$, established by equality

$$y^*(y) = \int_D (y(v)|d\lambda) = \sum_{i=1}^n \int_D y_i(v) d\lambda_i, \text{ where } y(\cdot) \in C^n(D)$$

and $\lambda = (\lambda_1, \dots, \lambda_n) \in \text{frm}(D)^n$ (see Theorem 1.5.8[5, p.138]).

Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \text{frm}(D)^n$, $\mu \in \text{frm}(D)$, $\lambda(E) = \int_E \varphi(v) d\mu + \lambda_s(E)$ Lebesgue expansion of the measure λ with concerning μ , where $\varphi(v) = (\varphi_1(v), \dots, \varphi_n(v)) \in L_1^n(D)$, $\lambda(E) = (\lambda_1(E_1), \dots, \lambda_n(E_n))$, $E = E_1 \times \dots \times E_n$, λ_s is a singular relative μ vector measure.

Let $R_{+\infty} = R \cup \{+\infty\}$ and $g: D \times R^n \rightarrow R_{+\infty}$ - normal integrand on $D \times R^n$. Let's consider functional $J(y) = \int_D g(v, y(v)) d\mu$ at the $y \in C^n(D)$ and put it $J^*(y_\lambda^*) = \sup_{y \in C^n(D)} \{ \int_D (y(t)|d\lambda) - J(y) \}$.

Subgradients J at the point $\bar{y} \in C^n(D)$ are definitions on the elements $y_\lambda^* \in C^n(D)^*$ for which

$$J(y) - J(\bar{y}) \geq y_\lambda^*(y - \bar{y}) \text{ at everyone } y \in C^n(D)$$

or that is equivalent to $J^*(y_\lambda^*) + J(\bar{y}) = y_\lambda^*(\bar{y})$. A bunch of everyone such subgradients are denoted by through $\partial J(\bar{y})$ and are called subdifferential functional J at the point $\bar{y}$.

Let's put $Q = \text{dom } J = \{y \in C^n(D) : J(y) < +\infty\}$.

Lemma 1. If g convex normal integrand on $D \times R^n$, exist $\bar{y} \in C^n(D)$ and $r > 0$ such that function $g(v, \bar{y}(v) + z)$ summable for $z \in R^n$, $\|z\| < r$, then

$$J^*(y_\lambda^*) = \int_D g^*(v, y^*(v)) d\mu + \delta_Q^*(y_{\lambda_s}^*),$$

where $\lambda(E) = \int_E y^*(v) d\mu + \lambda_s(E)$ Lebesgue decomposition of a measure λ relatively μ , $\delta_Q(y)$ - indicator function sets Q , $\delta_Q^*(y_{\lambda_s}^*) = \sup_{y \in Q} \int_D (y(v)|d\lambda_s)$.

Corollary 1. Let convex normal integrand on $D \times R^n$, $\bar{y} \in Q$ and $\text{int } Q \neq \emptyset$. Then $y_\lambda^* \in C^n(D)^*$, where $\lambda(E) = \int_E y^*(v) d\mu + \lambda_s(E)$, belongs the set $\partial J(\bar{y})$ in only that case when $y^*(v) \in \partial g(v, \bar{y}(v))$ at $v \in D$ and $\delta_Q^*(y_{\lambda_s}^*) = \int_D (\bar{y}(v)|d\lambda_s)$.

Note that if $\bar{y}(\cdot) \in \text{int } Q$ then $y_{\lambda_s}^* \equiv 0$. If Z is a dense subspace in $C^n(D)$, then by replacing $C^n(D)$ everywhere on Z it is easy to check that statements similar to lemma 1 and corollary 1 are also true in Z .

3. About subdifferentiability of the integral functional in $W_{p,1}^n[t_0, T]$

Symbol $W_{p,m}^n[t_0, T]$ denoted by Banach space of absolutely continuous together with its derivatives before order

$m-1$ inclusive mappings segment $[t_0, T]$ in R^n , the derivative in which belongs $L_p^n[t_0, T]$, where $-\infty < t_0 < T < +\infty$.

Let $\psi_1: [0, T] \times R^n \rightarrow R_{+\infty}$ and $\psi_2: [0, S] \times R^n \rightarrow R_{+\infty}$ be normal integrands. Let's put $J_{\psi_1}(u) = \int_0^T \psi_1(t, u(t, 0)) dt$ and $J_{\psi_2}(u) = \int_0^S \psi_2(s, u(0, s)) ds$ where $u \in A_p^n([0, T] \times [0, S])$ $1 \leq p < \infty$.

Lemma 2. If $v_1^* = (c_1, a_1(\cdot), b_1(\cdot), v_1(\cdot))$ and $v_2^* = (c_2, a_2(\cdot), b_2(\cdot), v_2(\cdot))$, that $J_{\psi_1}^*(v_1^*) < +\infty$ and $J_{\psi_2}^*(v_2^*) < +\infty$ then $b_1(s) = 0$, $v_1(t, s) = 0$ and $a_2(t) = 0$, $v_2(t, s) = 0$.

Thus, a study of the subdifferentiability of functionals $J_{\psi_1}(u) = \int_0^T \psi_1(t, u(t, 0)) dt$ and

$J_{\psi_2}(u) = \int_0^S \psi_2(s, u(0, s)) ds$ in $A_p^n([0, T] \times [0, S])$, $1 \leq p < \infty$ leads to a study of subdifferentiability of functionals $J_{\psi_1}(x) = \int_0^T \psi_1(t, x(t)) dt$ and $J_{\psi_2}(y) = \int_0^S \psi_2(s, y(s)) ds$ in spaces $W_{p,1}^n[0, T]$ and $W_{p,1}^n[0, S]$ respectively.

Let $-\infty < t_0 < T < +\infty$, $\psi: [t_0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}_{+\infty}$ and $J_{\psi}(x) = \int_{t_0}^T \psi(t, x(t)) dt$.

A bunch of $W_{p,1}^n[t_0, T]$ with the norm $\|x(\cdot)\| = \max_{t \in [t_0, T]} \|x(t)\|$ let's denote $C_{p,1}^n[t_0, T]$. It's clear that $C_{p,1}^n[t_0, T]^* \subset W_{p,1}^n[t_0, T]^*$.

Let $\varphi: W_{p,1}^n[t_0, T] \rightarrow \mathbb{R}_{+\infty}$ convex function. Subdifferential of a function φ at a point $\bar{x}$ relative to space $W_{p,1}^n[t_0, T]$ (relatively $C_{p,1}^n[t_0, T]$) denote by $\partial_W \varphi(\bar{x})$ ($\partial_C \varphi(\bar{x})$). Closure and interiority of a set M in space $W_{p,1}^n[t_0, T]$ ($C_{p,1}^n[t_0, T]$) denote accordingly $cl_W M$ and $int_W M$ ($cl_C M$, $int_C M$).

Lemma 3. If φ a lower semicontinuous proper sublinear function in $C_{p,1}^n[t_0, T]$, then $\partial_C \varphi(\bar{x}) = \partial_W \varphi(\bar{x})$ at $\bar{x} \in C_{p,1}^n[t_0, T]$.

Let $\varphi: C_{p,1}^n[t_0, T] \rightarrow \mathbb{R}_{+\infty}$ be a function. The closure in $C_{p,1}^n[t_0, T]$ of a function φ is a sign of $cl_C \varphi$ and the closure in $W_{p,1}^n[t_0, T]$ of a function φ is a sign of $cl_W \varphi$.

Lemma 4. If φ is proper convex, in particular sublinear in $C_{p,1}^n[t_0, T]$ function φ is continuous in $int_C \text{dom} \varphi$ and $int_C \text{dom} \varphi = int_W \text{dom} \varphi \neq \emptyset$, then $cl_C \varphi = cl_W \varphi$.

If Z normed space, $h: Z \rightarrow \mathbb{R}_{+\infty}$ convex function and $\bar{z} \in \text{dom} h$, then putting $h(\bar{z}; z) = \lim_{\lambda \downarrow 0} \frac{1}{\lambda} (h(\bar{z} + \lambda h) - h(\bar{z}))$.

Lemma 5. If φ proper convex functions in $C_{p,1}^n[t_0, T]$, $int_C \text{dom} \varphi = int_W \text{dom} \varphi \neq \emptyset$,

φ is continuous in $int_C \text{dom} \varphi$, $\partial_C \varphi(\bar{x})$ or $\partial_W \varphi(\bar{x})$ is non-empty, then $cl_C \varphi(\bar{x}; \cdot) = cl_W \varphi(\bar{x}; \cdot)$.

Lemma 6. If $\psi: [t_0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}_{+\infty}$ normal integrand, $\bar{x} \in L_p^n[t_0, T]$ and $J_{\psi}(\bar{x})$ is finite, then $J'_{\psi}(\bar{x}; x) = \int_{t_0}^T \psi'(t, \bar{x}(t); x(t)) dt$ at $x \in L_p^n[t_0, T]$.

If $\psi'(t, \bar{x}(t); \cdot)$ normal convex integrand, then $int_C \text{dom} J'_{\psi}(\bar{x}; \cdot) = int_W \text{dom} J'_{\psi}(\bar{x}; \cdot)$. Therefore, by lemma 4 we have that if $int_C \text{dom} J'_{\psi}(\bar{x}; \cdot)$ (or $int_W \text{dom} J'_{\psi}(\bar{x}; \cdot)$) is non-empty, then $\partial_C J_{\psi}(\bar{x}) = \partial_W J_{\psi}(\bar{x})$.

Lemma 7. Let ψ be a convex normal integrand, there exist function $x_0(\cdot) \in W_{p,1}^n[t_0, T]$ and $\varepsilon > 0$ so and so that function $\psi(t, x_0(t) + x)$ is summable at $x \in \mathbb{R}^n$, $\|x\| < \varepsilon$, where $\varepsilon > 0$, $\bar{x}(\cdot) \in W_{p,1}^n[t_0, T]$ and $cl_C J'_{\psi}(\bar{x}; \cdot)$ is proper functional, then $\partial_C J_{\psi}(\bar{x}) = \partial_W J_{\psi}(\bar{x})$.

4. About subdifferentiability integral functional in $A_p^n([0, T] \times [0, S])$

Let be $g: ([0, T] \times [0, S]) \times \mathbb{R}^n \rightarrow \mathbb{R}_{+\infty}$ proper convex function. Let's put

$$J_g(u(\cdot)) = \int_0^T \int_0^S g(t, s, u(t, s)) dt ds.$$

The subdifferential of a function $J_g(u(\cdot))$ at a point $\bar{u}$ relative to space $A_p^n([0, T] \times [0, S])$ (relative to space $C_p^n([0, T] \times [0, S])$) will denoted by $\partial_{A_p} J_g(\bar{u})$ ($\partial_{C_p} J_g(\bar{u})$). The remaining designations are also similar to section 3.

Lemma 8. If $\varphi: C_p^n([0, T] \times [0, S]) \rightarrow \mathbb{R}_{+\infty}$ is a lower semicontinuous proper sublinear function in $C_p^n([0, T] \times [0, S])$, then $\partial_{C_p} \varphi(0) = \partial_{A_p} \varphi(0)$.

Lemma 9. If φ is proper convex, in particular sublinear in $C_p^n([0, T] \times [0, S])$ function φ is continuous in $int_{C_p} \text{dom} \varphi$ and $int_{C_p} \text{dom} \varphi = int_{A_p} \text{dom} \varphi \neq \emptyset$, then $cl_{C_p} \varphi = cl_{A_p} \varphi$.

Lemma 10. If $g: [0, T] \times [0, S] \times \mathbb{R}^n \rightarrow \mathbb{R}_{+\infty}$ convex normal integrand, $\bar{u} \in L_p^n([0, T] \times [0, S])$ and $J_g(\bar{u})$ is

finite, then $J'_g(\bar{u}; u) = \int_0^T \int_0^S g'(t, s, \bar{u}(t, s); u(t, s)) dt ds$ at $u \in L_p^n([0, T] \times [0, S])$.

If $g'(t, s, \bar{u}(t, s); \cdot)$ normal convex integral, then $\text{int}_{C_p} \text{dom } J'_g(\bar{u}; \cdot) = \text{int}_{A_p} \text{dom } J'_g(\bar{u}; \cdot)$. Therefore if $\text{int}_{C_p} \text{dom } J'_g(\bar{u}; \cdot)$ (or $\text{int}_{A_p} \text{dom } J'_g(\bar{u}; \cdot)$) is not empty, then $\partial_{C_p} J_g(\bar{u}) = \partial_{A_p} J_g(\bar{u})$.

Lemma 11. If g is a convex normal integrand and in point $\bar{u}(t, s) \in A_p^n([0, T] \times [0, S])$ function $g(t, s, \bar{u}(t, s) + z)$ summable in $[0, T] \times [0, S]$ for $z \in \mathbb{R}^n$, $\|z\| < r$, where $r > 0$, then for each $v^* \in \partial_{A_p} J_g(\bar{u})$ there is a functional $y^* \in \partial_{C_p} J_g(\bar{u})$ such that $v^*(u) = y^*(u)$ for $u \in A_p^n([0, T] \times [0, S])$, where $y^*(u) = \int_0^T \int_0^S (u(t, s) | \omega(t, s)) dt ds$, $\omega(t, s) \in \partial g(t, s, \bar{u}(t, s))$ and $\omega \in L_1^n([0, T] \times [0, S])$.

Lemma 12. Let g is a convex normal integrand, there exists $u_0(\cdot) \in A_p^n([0, T] \times [0, S])$ and $\varepsilon > 0$ such that function $g(t, s, u_0(t, s) + z)$ is summable at $z \in \mathbb{R}^n$, $\|z\| < \varepsilon$, where $\varepsilon > 0$, $\bar{u}(\cdot) \in A_p^n([0, T] \times [0, S])$ and $u \rightarrow \text{cl}_{C_p} J'_g(\bar{u}; u)$ is proper functional, then $\partial_{C_p} J(\bar{u}) = \partial_{A_p} J(\bar{u})$.

5. Subdifferential of terminal functional in $A_p^n([0, T] \times [0, S])$

Let $1 \leq p < +\infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. Let's put it $J_q(u) = q(u(0, 0), u(T, 0), u(0, S), u(T, S))$.

Lemma 13. If $v^* = (c, a(\cdot), b(\cdot), v(\cdot))$ and $J_q(v^*) < +\infty$, then there exists a vector $(d_1, d_2, d) \in \mathbb{R}^{3n}$ such that $a(t) = d_1 \in \mathbb{R}^n$, $b(t) = d_2 \in \mathbb{R}^n$, $v(t, s) = d \in \mathbb{R}^n$ and

$$J_q(v^*) = q^*(c - d_1 - d_2 + d, d_1 - d, d_2 - d, d).$$

Corollary 2. If q is a convex function and $v^* = (c, a(\cdot), b(\cdot), v(\cdot)) \in \partial J_q(\bar{u})$, then there exists a vector

$(d_1, d_2, d) \in \mathbb{R}^{3n}$ such that $a(t) = d_1 \in \mathbb{R}^n$, $b(t) = d_2 \in \mathbb{R}^n$, $v(t, s) = d \in \mathbb{R}^n$ and

$$(c - d_1 - d_2 + d, d_1 - d, d_2 - d, d) \in \partial q(\bar{u}(0, 0), \bar{u}(T, 0), \bar{u}(0, S), \bar{u}(T, S)).$$

6. On the subdifferentiability of sums of integral functional in $A_p^n([0, T] \times [0, S])$

Let $g : [0, T] \times [0, S] \times \mathbb{R}^n \rightarrow \mathbb{R}_{+\infty}$,

$$\psi_1 : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}_{+\infty}, \quad \psi_2 : [0, S] \times \mathbb{R}^n \rightarrow \mathbb{R}_{+\infty}$$

normal integrands, $q : \mathbb{R}^{4n} \rightarrow \mathbb{R}_{+\infty}$.

Let us consider the subdifferentiability of the functional

$$J_p(u) = \int_0^T \int_0^S g(t, s, u(t, s)) dt ds + \int_0^T \psi_1(t, u(t, 0)) dt + \int_0^S \psi_2(s, u(0, s)) dt + q(u(0, 0), u(T, 0), u(0, S), u(T, S))$$

in the $A_p^n([0, T] \times [0, S])$ space, where $1 \leq p < +\infty$.

Let's put

$$F_0(u) = \int_0^T \int_0^S g(t, s, u(t, s)) dt ds, \quad F_1(u) = \int_0^T \psi_1(t, u(t, 0)) dt, \quad F_2(u) = \int_0^S \psi_2(s, u(0, s)) dt, \quad F_3(u) = q(u(0, 0), u(T, 0), u(0, S), u(T, S))$$

$$Q = \{u \in C_p^n([0, T] \times [0, S]) : F_0(u) < +\infty\},$$

$$Q_1 = \{x \in C_{p,1}^n[0, T] : F_1(x) < +\infty\},$$

$$Q_2 = \{y \in C_{p,1}^n[0, S] : F_2(y) < +\infty\}.$$

Theorem 1. Let $g : [0, T] \times [0, S] \times \mathbb{R}^n \rightarrow \mathbb{R}_{+\infty}$, $\psi_1 : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}_{+\infty}$, $\psi_2 : [0, S] \times \mathbb{R}^n \rightarrow \mathbb{R}_{+\infty}$ be convex normal integrands, $q : \mathbb{R}^{4n} \rightarrow \mathbb{R}_{+\infty}$ is a convex function, there exists a function $u_0 \in A_p^n([0, T] \times [0, S])$ and a number $r > 0$ such that the functions $g(t, s, u_0(t, s) + y)$, $\psi_1(t, u_0(t, 0) + y)$, $\psi_2(s, u_0(0, s) + y)$ are summable at $y \in \mathbb{R}^n$, $\|y\| < r$ and $q(u_0(0, 0), u_0(T, 0), u_0(0, S), u_0(T, S))$ finitely, $\bar{u} \in A_p^n([0, T] \times [0, S])$.

Then $v^* = (c, a(\cdot), b(\cdot), v(\cdot)) \in \partial J_p(\bar{u})$ if and only if there exist measures $\lambda \in \text{frm}([0, T] \times [0, S])^n$, $\mu \in \text{frm}[0, T]^n$, $\gamma \in \text{frm}[0, S]^n$ and $(\bar{c}, d_1, d_2, d) \in \mathbb{R}^{4n}$ such that

$$1) \quad v^*(u) = (\bar{c} - d_1 - d_2 + d | \mu(0, 0)) + (d_1 - d | \mu(T, 0)) + (d_2 - d | \gamma(0, S)) + (d | \gamma(T, S)) +$$

$$+ \int_0^T \int_0^S (u(t, s)) d\lambda_s + \int_0^T (u(t, 0)) d\mu + \int_0^S (u(0, s)) d\gamma \quad \text{at}$$

$$u \in A_p^n([0, T] \times [0, S]),$$

$$2) \quad \omega(t, s) \in \partial g(t, s, \bar{u}(t, s)),$$

$$3) \quad \omega_1(t) \in \partial \psi_1(t, \bar{u}(t, 0)),$$

$$4) \quad \omega_2(s) \in \partial \psi_2(s, \bar{u}(0, s)),$$

$$5) \quad (\bar{c} - d_1 - d_2 + d, d_1 - d, d_2 - d, d) \in \partial q(\bar{u}(0, 0), \bar{u}(T, 0), \bar{u}(0, S), \bar{u}(T, S)),$$

$$6) \quad \max_{u \in Q} \int_0^T \int_0^S (u(t, s)) d\lambda_s = \int_0^T \int_0^S (\bar{u}(t, s)) d\lambda_s,$$

$$7) \quad \max_{x \in Q_1} \int_0^T (x(t)) d\mu_s = \int_0^T (\bar{u}(t, 0)) d\mu_s,$$

$$8) \quad \max_{y \in Q_2} \int_0^S (y(s)) d\gamma_s = \int_0^S (\bar{u}(0, s)) d\gamma_s,$$

$$\text{where } \lambda(E) = \iint_E \omega(t, s) dt ds + \lambda_s(E),$$

$$\mu(E_1) = \int_{E_1} \omega_1(t) dt + \mu_s(E_1),$$

$$\gamma(E_2) = \int_{E_2} \omega_2(s) ds + \gamma_s(E_2)$$

Lebeg decomposition λ, μ and ν respectively.

Theorem 2. Let $g: [0, T] \times [0, S] \times \mathbb{R}^n \rightarrow \mathbb{R}_{+\infty}$, $\psi_1: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}_{+\infty}$, $\psi_2: [0, S] \times \mathbb{R}^n \rightarrow \mathbb{R}_{+\infty}$ be convex normal integrands, $q: \mathbb{R}^{4n} \rightarrow \mathbb{R}_{+\infty}$ a convex function, there is a number $r > 0$ such that the functions $g(t, s, \bar{u}(t, s) + y)$, $\psi_1(t, \bar{u}(t, 0) + y)$, $\psi_2(s, \bar{u}(0, s) + y)$ are summable at $y \in \mathbb{R}^n$, $\|y\| < r$ and the number $q(\bar{u}(0, 0), \bar{u}(T, 0), \bar{u}(0, S), \bar{u}(T, S))$ is finite. Then $v^* = (c, a(\cdot), b(\cdot), v(\cdot)) \in \partial J_p(\bar{u})$ if and only if $a(\cdot) \in W_{1,1}^n[0, T]$, $b(\cdot) \in W_{1,1}^n[0, S]$, $v(\cdot) \in A_1^n([0, T] \times [0, S])$ and $v(T, s) = v(t, S) = \text{const}$ for $t \in [0, T]$, $s \in [0, S]$ such that

$$1) \quad v_s(t, s) \in \partial g(t, s, \bar{u}(t, s)),$$

$$2) \quad v_t(t, 0) - \dot{a}(t) \in \partial \psi_1(t, \bar{u}(t, 0)),$$

$$3) \quad v_s(0, s) - \dot{b}(s) \in \partial \psi_2(s, \bar{u}(0, s)),$$

$$4) \quad (c - a(0) - b(0) + v(0, 0), a(T) - v(T, 0), b(S) - v(0, S), v(T, S)) \in \partial q(\bar{u}(0, 0), \bar{u}(T, 0), \bar{u}(0, S), \bar{u}(T, S))$$

7. About subdifferential non-convex functional

Let $f: [0, T] \times [0, S] \times \mathbb{R}^n \rightarrow \mathbb{R}_{+\infty}$, $\varphi_1: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}_{+\infty}$, $\varphi_2: [0, S] \times \mathbb{R}^n \rightarrow \mathbb{R}_{+\infty}$ $L \times B$ -measurable integrands, $q: \mathbb{R}^{4n} \rightarrow \mathbb{R}_{+\infty}$. Let us consider the subdifferentiability of a non-convex functional

$$I_p(u) = \int_0^T \int_0^S f(t, s, u(t, s)) dt ds + \int_0^T \varphi_1(t, u(t, 0)) dt + \int_0^S \varphi_2(s, u(0, s)) ds + q(u(0, 0), u(T, 0), u(0, S), u(T, S))$$

in space $A_p^n([0, T] \times [0, S])$ where $1 \leq p < +\infty$.

Lemma 14. If functions $(t, s) \rightarrow f(t, s, z)$, $t \rightarrow \varphi_1(t, u_1)$, $s \rightarrow \varphi_2(s, v_1)$ measurable, exist functions $k(\cdot) \in L_1([0, T] \times [0, S])$,

$$k_1(\cdot) \in L_1[0, T], \quad k_2(\cdot) \in L_1[0, S],$$

where $k(t, s) \geq 0$, $k_1(t) \geq 0$, $k_2(s) \geq 0$, numbers $\lambda > 0$ and $\alpha > 0$ such that $|f(t, s, v) - f(t, s, w)| \leq k(t, s) \|v - w\|$ at $\|v - \bar{u}(t, s)\| \leq \alpha$, $\|w - \bar{u}(t, s)\| \leq \alpha$, $|\varphi_1(t, x) - \varphi_1(t, x_1)| \leq k_1(t) \|x - x_1\|$ at $\|x - \bar{u}(t, 0)\| \leq \alpha$, $\|x_1 - \bar{u}(t, 0)\| \leq \alpha$,

$$|\varphi_2(t, \tilde{x}) - \varphi_2(t, \tilde{x}_1)| \leq k_2(s) \|\tilde{x} - \tilde{x}_1\| \quad \text{at } \|\tilde{x} - \bar{u}(0, s)\| \leq \alpha, \|\tilde{x}_1 - \bar{u}(0, s)\| \leq \alpha,$$

$$|q(u_1, u_2, u_3, u_4) - \varphi_i(z_1, z_2, z_3, z_4)| \leq \lambda (\|u_1 - z_1\| + \|u_2 - z_2\| + \|u_3 - z_3\| + \|u_4 - z_4\|)$$

at $\|u_1 - \bar{u}(0, 0)\| \leq \alpha$, $\|z_1 - \bar{u}(0, 0)\| \leq \alpha$, $\|u_2 - \bar{u}(T, 0)\| \leq \alpha$, $\|z_2 - \bar{u}(T, 0)\| \leq \alpha$, $\|u_3 - \bar{u}(0, S)\| \leq \alpha$, $\|z_3 - \bar{u}(0, S)\| \leq \alpha$, $\|u_4 - \bar{u}(T, S)\| \leq \alpha$, $\|z_4 - \bar{u}(T, S)\| \leq \alpha$ and $I_p(\bar{u})$ is finite, then functional $I_p(u)$ satisfies the condition Lipschitz in neighborhood points $\bar{u}(\cdot)$ in $A_p^n([0, T] \times [0, S])$.

Let be X a Banach space, $g: X \rightarrow \mathbb{R}_{+\infty}$, $x_0 \in \text{dom } g$. If g is the Lipschitz function is near x_0 , then let us put (see [8, p.32])

$$g^{[1]}(x_0; x) = \limsup_{y \rightarrow x_0, \lambda \downarrow 0} \frac{g(y + \lambda x) - g(y)}{\lambda},$$

$$\partial_C g(x_0) = \{x^* \in X^* : g^{[1]}(x_0; x) \geq \langle x^*, x \rangle \text{ at } x \in X\}.$$

Theorem 3. If the conditions of lemma 14 are satisfied and $v^* = (c, a(\cdot), b(\cdot), v(\cdot)) \in \partial_C I_p(\bar{u})$, then $v(\cdot) \in A_1^n([0, T] \times [0, S])$, $a(\cdot) \in W_{1,1}^n[0, T]$, $b(\cdot) \in W_{1,1}^n[0, S]$ and $v(T, s) = v(t, S) = \text{const}$ at $t \in [0, T]$, $s \in [0, S]$ and such that $v_{ts}(t, s) \in \partial_C f(t, s, \bar{u}(t, s))$, $v_t(t, 0) - \dot{a}(t) \in \partial_C \varphi_1(t, \bar{u}(t, 0))$, $v_s(0, s) - \dot{b}(s) \in \partial_C \varphi_2(s, \bar{u}(0, s))$, $(c - a(0) - b(0) + v(0, 0), a(T) - v(T, 0), b(S) - v(0, S), v(T, S)) \in \partial_C q(\bar{u}(0, 0), \bar{u}(T, 0), \bar{u}(0, S), \bar{u}(T, S))$.

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İKİDƏYİŞƏNLİ MÜTLƏQ KƏSİLMƏZ FUNKSIYALAR SİNFİNDƏ FUNKSIONALIN SUBDİFERENSİALI HAQQINDA

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İşdə kəsilməz funksiyalar fəzasında, mütləq kəsilməz və ikiölçülü mütləq kəsilməz funksiyalar fəzasında xüsusi formalı funksionalın subdiferensialını öyrənir. Mütləq kəsilməz və ikiölçülü mütləq kəsilməz funksiyalar fəzasında iki müxtəlif normaya baxılır və bu normalardakı subdiferensialların üst-üstə düşdüyü göstərilir. Bu bərabərliklərdən istifadə edərək müxtəlif integral funksiyaların subdiferensialları öyrənilir. Sonra ikiölçülü mütləq kəsilməz funksiyalar sinfində klassik variasiya hesabı lemmasının analoqu isbat olunur. Bu lemmadan istifadə edərək ikiölçülü mütləq kəsilməz funksiyalar sinfində terminal funksionalın subdiferensialı öyrənilir.

Açar sözlər: subdiferensial, integral funksional, funksiya, çoxluq

О СУБДИФФЕРЕНЦИАЛЕ ФУНКЦИОНАЛА В ДВУМЕРНЫХ АБСОЛЮТНО НЕПРЕРЫВНЫХ ФУНКЦИЯХ

М.А.Садыгов

В работе изучается субдифференциал функционала специального вида в пространстве непрерывных функций, в пространстве абсолютно непрерывных и двумерных абсолютно непрерывных функций. В пространстве абсолютно непрерывных и двумерных абсолютно непрерывных функций рассматриваются две различные нормы и показано, что субдифференциалы в этих нормах совпадают. Используя эти равенства, исследуются субдифференциалы различных интегральных функционалов. Далее доказывается аналог леммы классического вариационного исчисления для двумерных абсолютно непрерывных функций. Используя эту лемму, изучается субдифференциал терминального функционала в классе двумерных абсолютно непрерывных функций.

Ключевые слова: субдифференциал, интегральный функционал, функция, множество