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THE INVERSE SCATTERING PROBLEM FOR A DISCRETE DIRAC OPERATOR ON THE ENTIRE LINE

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A discrete analogue of the one-dimensional Dirac system on the whole line is considered. The coefficients of the system are steplike. Using the transformation operators special solutions with asymptotic behavior at infinity are constructed. The inverse scattering problem is studied. We derive a necessary and a sufficient condition on the scattering data so that the inverse problem is uniquely solvable.

Keywords: *discrete Dirac operator, scattering problem, transformation operators, inverse problem, main equations*

We consider the system of difference equations

$$\begin{cases} a_{1,n}y_{2,n+1} + a_{2,n}y_{2,n} = \lambda y_{1,n}, \\ a_{1,n-1}y_{1,n-1} + a_{2,n}y_{1,n} = \lambda y_{2,n}, \ n = 0,\pm1,\pm2,\ldots, \end{cases} \quad (1)$$

where the coefficients $a_{1,n}, a_{2,n}$ satisfy the conditions

$$\left.\begin{array}{l} a_{1,n} > 0,\ a_{2,n} < 0,\ n = 0,\pm1,\pm2,..., \\ \sum\limits_{n\geq1}|n|\left\{|a_{1,n} - A| + |a_{2,n} + A|\right\} + \sum\limits_{n\leq-1}|n|\left\{|a_{1,n} - 1| + |a_{2,n} + 1|\right\} < \infty \end{array}\right\}, \quad (2)$$

with $A > 0$. The system of equations (1) is difference analog of the one-dimensional Dirac system, for which the inverse scattering problem has been studied in $[1]-[2]$. On the other hand, the direct and inverse scattering problems for the system of equations (1) were investigated in $[3]-[4]$ with $A = 1$.

In this work we study the inverse scattering problem the system of difference equations (1) with coefficients from (2). We derive conditions on the scattering data which are necessary and sufficient for the existence of unique coefficients $a_{j,n} > 0,\ j = 1,2$, in the class (2). Availability of step-like coefficients brings to the change of the properties of the reflection coefficient. For these reasons, some classical arguments needed to be modified considerably on obtaining main equation.

Note that similar problem for the one-dimensional Dirac system and Schrodinger equation and their discrete analogue was considered in $[5]-[6]$. In $[7]$, some spectral properties of the other analogue of the one-dimensional Dirac system were investigated.

For definiteness, it is assumed hereinafter that $A \leq 1$. Let Γ_j denote the complex λ-plane cut along the segment $\left[-2A^{2-j}, 2A^{2-j}\right], j = 1,2$ and $\partial\Gamma_j$ denote its boundary. In Γ_j, consider the function

$$z_j = z_j(\lambda) = -\frac{\lambda^2 - 2A^{2(2-j)}}{2A^{2(2-j)}} + \frac{\lambda}{2A^{2-j}}\sqrt{\lambda^2 - 4A^{2(2-j)}},$$

where the regular branch of the radical is determined by the condition $\sqrt{\lambda^2 - 4A^{2(2-j)}} > 0$ at $\lambda > 2A^{2-j}, j = 1,2$.

Theorem 1. *The system of equations (1) has solutions, which can be represented in the form*

$$f_{j,n}(\lambda) = \alpha_j^+(n)\left(\frac{z_1 - 1}{A^{-1}\lambda}\right)^{2-j} \times$$
$$\times z_1^n\left(1 + \sum_{m\geq1} K_j^+(n,m) z_1^m\right)$$

$$g_{j,n}(\lambda)=\alpha_j^-(n)\left(\frac{z_2^{-1}-1}{\lambda}\right)^{2-j}\times \times z_2^{-n}\left(1+\sum_{m\le -1}K_j^-(n,m)z_2^{-m}\right), n\in Z, \tag{3}$$

where the quantities $\alpha_1^\pm(n),\alpha_2^\pm(n),K_1^\pm(n,m),\ K_2^\pm(n,m)$ *satisfy the relations*

$$\left.\begin{array}{l}\alpha_j^\pm(n)=1+o(1)\ \ at\ n\to\pm\infty,\ j=1,2,\\ K_j^\pm(n,m)=O\left(\sigma^\pm\left(n+\left[\frac{m}{2}\right]+\frac{1\mp1}{2}\right)\right),\\ n+m\to\pm\infty\end{array}\right\} \tag{4}$$

where

$$\sigma^\pm(n)=\sum_{\pm m\ge \pm n}\left\{\left|a_{1,m}-A^{\frac{1\pm1}{2}}\right|+\left|a_{2,m}+A^{\frac{1\pm1}{2}}\right|\right\},\ and$$

$[\cdot]$ *denotes integer part.*

Moreover, we have

$$\left.\begin{array}{l}\dfrac{a_{1,n}}{A^{\frac{1\pm1}{2}}}=\left(\dfrac{\alpha_2^\pm(n+1)}{\alpha_1^\pm(n)}\right)^{\pm1},\ \dfrac{a_{2,n}}{A^{\frac{1\pm1}{2}}}=-\left(\dfrac{\alpha_1^\pm(n)}{\alpha_2^\pm(n)}\right)^{\pm1},\\ \dfrac{a_{1,n}^2-A^{1\pm1}}{A^{1\pm1}}=\pm K_2^\pm\left(n+\dfrac{1\mp1}{2},\pm1\right)\mp\\ K_1^\pm\left(n+\dfrac{1\mp1}{2},\pm1\right),\\ \dfrac{a_{2,n}^2-A^{1\pm1}}{A^{1\pm1}}=\pm K_1^\pm\left(n-\dfrac{1\pm1}{2},\pm1\right)\mp\\ K_2^\pm\left(n+\dfrac{1\mp1}{2},\pm1\right),\ n=0,\pm1,...,\end{array}\right\} \tag{5}$$

According to (3), (4), for any n the solutions $\{f_{j,n}(\lambda)\}$ и $\{g_{j,n}(\lambda)\}$, $j=1,2$, that is regular on Γ_1 and Γ_2, is continuous up to its boundary $\partial\Gamma_1$ and $\partial\Gamma_2$ appropriately.

We also expansion

$$g_{j,n}(\lambda)=a_1(\lambda)\overline{f_{j,n}(\lambda)}+b_1(\lambda)f_{j,n}(\lambda), \lambda\in\partial\Gamma_1, \lambda^2\ne 4A^2, \tag{6}$$

$$f_{j,n}(\lambda)=a_2(\lambda)\overline{g_{j,n}(\lambda)}+b_2(\lambda)g_{j,n}(\lambda), \lambda\in\partial\Gamma_2, \lambda^2\ne 4. \tag{7}$$

Here, $a_j(\lambda),b_j(\lambda),\ j=1,2$, are defined by the formulas

$$\begin{array}{l}a_j(\lambda)=\dfrac{\lambda W\left[f_{j,n}(\lambda),g_{j,n}(\lambda)\right]}{A^{2(2-j)}\left(z_j-z_j^{-1}\right)},\\ b_1(\lambda)=\dfrac{\lambda W\left[\overline{f_{j,n}(\lambda)},g_{j,n}(\lambda)\right]}{A^2\left(z_1^{-1}-z_1\right)},\\ b_2(\lambda)=\dfrac{\lambda W\left[f_{j,n}(\lambda),\overline{g_{j,n}(\lambda)}\right]}{z_2^{-1}-z_2},\end{array} \tag{8}$$

where $W\left[f_{j,n},g_{j,n}\right]=a_{1,n}\left(f_{1,n-1}g_{2,n}-f_{2,n}g_{1,n-1}\right)$ is the Wronskian of the solutions $\{f_{j,n}\}$ and $\{g_{j,n}\}$. The basic properties of the coefficients $a_j(\lambda),b_j(\lambda)\ j=1,2$, are established using (6)-(8). They can be rewritten as the following condition.

I. *The functions* $a_2^{-1}(\lambda),b_2(\lambda)$ *are continuous for* $\lambda\in\partial\Gamma_2$ *and* $a_2^{-1}(\lambda)$ *can be regularly extended to* Γ_2 *, except for possibly simple poles* $\lambda_k=\pm\mu_k,\ \mu_k>0,\ k=1,...,N$ *, lying on the real line outside the cut* $[-2,2]$. *It is true that* $\lim_{\lambda\to\infty}a_2(\lambda)=d>0$ *and*

$$\left.\begin{array}{c}a_j(\lambda-i0)=\overline{a_j(\lambda+i0)},\\ b_j(\lambda-i0)=\overline{b_j(\lambda+i0)}\\ b_2(\lambda)=\overline{a_2(\lambda)}, \lambda\in\partial\Gamma_2\setminus\partial\Gamma_1,\\ A^2\left(z_1^{-1}-z_1\right)b_1(\lambda)=-\left(z_2^{-1}-z_2\right)\overline{b_2(\lambda)}, \lambda\in\partial\Gamma_1\\ A^2\left(z_1^{-1}-z_1\right)a_1(\lambda)=\left(z_2^{-1}-z_2\right)a_2(\lambda), \lambda\in\Gamma_2\\ \left|a_j(\lambda)\right|^2-\left|b_j(\lambda)\right|^2=\left(\dfrac{A^2\left(z_1^{-1}-z_1\right)}{z_2^{-1}-z_2}\right)^{(-1)^j},\\ j=1,2, \lambda\in\partial\Gamma_1\end{array}\right\}$$

Let

$$\left(m_k^+\right)^{-2}=\sum_{n\in Z}\left\{f_{1,n}^2(\pm\mu_k)+f_{1,n}^2(\pm\mu_k)\right\},$$

$$\left(m_k^-\right)^{-2} = \sum_{n\in Z}\left\{g_{1,n}^2\left(\pm\mu_k\right)+g_{1,n}^2\left(\pm\mu_k\right)\right\}.$$

It is easy to verify that

$$m_k^+ m_k^- = \left(a_j'(\lambda)\frac{A^{2(2-j)}\left(z_j - z_j^{-1}\right)}{\lambda}\Bigg|_{\lambda=\lambda_k}\right)^{-2},$$

$k = 1,..., N,\ j = 1,2.$

The set of quantities $\left\{a_2(\lambda), b_2(\lambda)\ \lambda\in\partial\Gamma_2; \pm\mu_k; m_k^- > 0, k = 1,..., N\right\}$ is called the scattering data for the system of equations (1). The inverse scattering problem for system of equations (1) is understood as the reconstruction of the coefficients $a_{1,n}, a_{2,n}$. In solving the inverse problem, an important role is played by the co-called Marchenko-type main equations. Let $\partial\Gamma_j^+$ denote the set of the points in the upper cut along segment $\left[-2A^{2-j}, 2A^{2-j}\right], j = 1,2$ in the λ-plane. Define

$$F_j^+(n) = \sum_{k=1}^{N}\left(m_k^+\right)^2\frac{\lambda z_1^n\left(\pm\mu_k\right)}{A^2\left(z_1^{-1}-z_1\right)}\left(\frac{Az_1 - A}{\lambda}\right)^{2(2-j)} + \frac{1}{2\pi i}\int_{\partial\Gamma_1}\frac{\lambda b_1(\lambda)}{A^2\left(z_1^{-1}-z_1\right)a_1(\lambda)}\left(\frac{Az_1 - A}{\lambda}\right)^{2(2-j)} z_1^n d\lambda + \frac{1}{2\pi i}\int_{\partial\Gamma_2^+\setminus\partial\Gamma_1^+}\frac{\lambda\left|a_2(\lambda)\right|^{-2}}{z_2^{-1}-z_2}\left(\frac{Az_1 - A}{\lambda}\right)^{2(2-j)} z_1^n d\lambda,$$

$$F_j^-(n) = \sum_{k=1}^{N}\left(m_k^-\right)^2\left(\frac{z_2^{-1}-1}{\lambda}\right)^{2(2-j)} z_2^{-n}\left(\pm\mu_k\right) + \frac{1}{2\pi i}\int_{\partial\Gamma_2}\frac{\lambda b_2(\lambda)}{\left(z_2^{-1}-z_2\right)a_2(\lambda)}\left(\frac{z_2^{-1}-1}{\lambda}\right)^{2(2-j)} z_2^{-n} d\lambda.$$

Theorem 2. *For any $n, n\in Z$, quantities $K_j^{\pm}(n,m), \alpha_j^{\pm}(n)$ involved in (3) satisfy the relations*

$$K_j^{\pm}(n,m) + F_j^{\pm}(2n+m) + \sum_{\pm r\geq 1}K_j^{\pm}(n,r)F_j^{\pm}(2n+m+r) = 0, \qquad (9)$$

$\pm m \geq 1,\ j = 1,2$

$$\left(\alpha_j^{\pm}(n)\right)^{-2} = 1 + F_j^{\pm}(2n) + \sum_{\pm r\geq 1}K_j^{\pm}(n,r)F_j^{\pm}(2n+r). \qquad (10)$$

The equations (9) are the main equations of the inverse problem. Applying research techniques available for main equations [3], [4], we can obtain the following necessary condition on the scattering data.

II. *Its holds that*

$$\sum_{\pm n\geq 1}|n|\left|F_1^{\pm}(n) - F_2^{\pm}(n)\right| < \infty,$$
$$\sum_{\pm n\geq 1}|n|\left|F_1^{\pm}(n+2) - F_1^{\pm}(n)\right| < \infty.$$

Theorem 3. *A set of quantities $\left\{a_2(\lambda), b_2(\lambda)\ \lambda\in\partial\Gamma_2; \pm\mu_k; m_k^- > 0, k = 1,..., N\right\}$ is scattering data for the system of Equations (1) with coefficients satisfying assumptions (2) if and only if conditions* **I–II** holds.

The reconstruct the system of equations (1), we consider main equations, which are constructed from the scattering data. Find $K_j^{\pm}(n,m), \alpha_j^{\pm}(n)$ from equations (9) and (10), respectively, the first one having a unique solution with respect to $K_j^{\pm}(n,m)$ under the conditions theorem 3. The coefficients $a_{1,n}, a_{2,n}$ can now be determined from any formulas (5).

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BÜTÜN OXDA DİSKRET DİRAK OPERATORU ÜÇÜN SƏPİLMƏNİN TƏRS MƏSƏLƏSİ

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Birölçülü Dirak sisteminin diskret analoqu baxılmışdır. Əmsallar pilləvarı tipdir. Çevirmə operatorlarının köməyilə sonsuzluqda şərt ödəyən xüsusi həllər qurulmuşdur. Səpilmənin tərs məsələsi öyrənilmişdir. Tərs məsələnin birqiymətli həlli üçün səpilmə verilənləri üzərinə zəruri və kafi şərtlər tapılmışdır.

Açar sözlər:*diskret Dirak operatoru, səpilmə məsələsi, çevirmə operatoru, tərs məsələ, əsas tənliklər*

ОБРАТНАЯ ЗАДАЧА РАССЕЯНИЯ ДЛЯ ДИСКРЕТНОГО ОПЕРАТОРА ДИРАКА НА ВСЕЙ ОСИ

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Рассмотрен дискретный аналог одномерной системы Дирака. Коэффициенты являются ступенеобразными. С помощью операторов преобразования построены специальные решения с условиями на бесконечности. Изучена обратная задача рассеяния. Получены необходимое и достаточное условия на данные рассеяния для однозначной разрешимости обратной задачи.

Ключевые слова: *дискретный оператор Дирака, задача рассеяния, оператор преобразования, обратная задача, основные уравнения*