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THE INVERSE SCATTERING PROBLEM FOR A SCHRÖDINGER EQUATION WITH AN ADDITIONAL LINEAR POTENTIAL

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A Schrödinger equation with an additional linear potential on the whole line is considered. Using the transformation operators special solutions with asymptotic behavior at infinity are constructed. The inverse scattering problem is studied. The main equation of inverse problem is obtained. We develop an algorithm for solving the inverse problem of reconstruction of perturbed coefficient.

Keywords: *the Schrödinger equation, transformation operators, inverse scattering problem, main equations*

Consider the differential equation

$$-y'' + [x + p(x)]y = \lambda y, \qquad -\infty < x < \infty, \quad (1)$$

where $p(x)$ is a real and is continuously-differentiable function on a whole axis and satisfy the condition

$$\int_{-\infty}^{\infty} (1+|x|)|p(x)|dx < \infty . \quad (2)$$

In this work we study the inverse scattering problem the differential equation difference with coefficient from (2). The existence of the transformation operators that takes the solution of the non-perturbed equation

$$-y'' + xy = \lambda y \quad (3)$$

to the solution of the perturbed equation (1). Using the transformation operator method, we give a derivation of the so-called main equation and prove its unique solvability. We provide an algorithm for the solution of the inverse scattering problem.

In the absence of linear potential similar problem for the differential equation (1) were investigated in [1]–[3]. The additional linear potential will require significant modification of the classic arguments of the works [1]–[3].

Note that the some inverse problems for equation (1) were investigated in the works [4], [5]. The scattering problem for the Schrödinger equation with an additional anharmonic potential is studied in the works [6].

In the complex λ -plane consider the function $k(\lambda, x) = \sqrt{\lambda - x}$, where the regular branch of the radical is determined by the condition $\sqrt{x-\lambda} > 0$ at $\lambda < x$. We can prove that the functions [7] (see, also [8])

$$f_0^+(x,\lambda) = \sqrt{\pi}\, Ai(x-\lambda),$$
$$f_0^-(x,\lambda) = \sqrt{\pi}\,[Ai(x-\lambda) - Bi(x-\lambda)],$$

are the solutions of equation (3), where $Ai(z)$ and $Bi(z)$ are the Airy functions of the first and second kind, respectively. We have the following formulas for expansion

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f_0^+(x,\lambda) f_0^-(y,\lambda)d\lambda = \delta(x-y),$$
$$\frac{1}{\pi}\int_{-\infty}^{\infty} Ai(x-\lambda)Ai(y-\lambda)d\lambda = \delta(x-y), \quad (5)$$

where δ - the Dirac delta function.

Denote

$$\sigma^{\pm}(x) = \pm\int_x^{\pm\infty} |p(x)|(x)dx , \; \sigma_1^{\pm}(x) = \pm\int_x^{\pm\infty} \sigma^{\pm}(x)dx .$$

The following theorem introduces the Jost solutions $f_+(x,\lambda)$ and $f_-(x,\lambda)$ with prescribed behavior in $\pm\infty$.

Theorem 1. *The equation (1) for all* λ, $\operatorname{Im}\lambda \ge 0$ *has unique solutions* $f_\pm(x,\lambda)$ *in the form*

$$f_\pm(x,\lambda) = f_0^\pm(x,\lambda) \pm \int_x^{\pm\infty} K^\pm(x,t) f_0^\pm(t,\lambda)dt, \quad (6)$$

where $K^\pm(x,t)$ *are real continuous functions, and*

$$\left|K^\pm(x,t)\right| \le \frac{1}{2}\sigma^\pm\left(\frac{x+t}{2}\right)e^{\sigma_1^\pm\left(\frac{x+t}{2}\right)}, \quad (7)$$

$$K^\pm(x,x) = \pm\frac{1}{2}\int_x^{\pm\infty} p(t)dt. \quad (8)$$

Consider the non perturbed equation (3). Clearly, for all real λ the solutions of equation (1) are also $\overline{f_0^+(x,\lambda)}$ and $\overline{f_0^-(x,\lambda)}$. On the other hand, it is known that (see, [8]) the functions $Ai(z), Bi(z)$ are linearly independent for all z and their Wronskian $W[\operatorname{Ai}(z),\operatorname{Bi}(z)]$ is $\frac{1}{\pi}$. Hence, are valid relations $W\left[f_0^-(x,\lambda), \overline{f_0^-(x,\lambda)}\right] = 2i$. From the formulas (3)-(5) it follows that the functions $\left\{f_-(x,\lambda), \overline{f_-(x,\lambda)}\right\}$ form fundamental systems of solutions for equation (1), and $W\left[f_-(x,\lambda), \overline{f_-(x,\lambda)}\right] = 2i$.

Therefore, we have for all real λ:

$$f_+(x,\lambda) = a(\lambda)\overline{f_-(x,\lambda)} + \overline{a(\lambda)}f_-(x,\lambda), \quad (9)$$

where

$$a(\lambda) = \frac{W[f_-(x,\lambda), f_+(x,\lambda)]}{2i}. \quad (10)$$

We note that (9), (10) gives the analytic continuation for $a(\lambda)$ to $\operatorname{Im}\lambda > 0$. Hence, the function $a(\lambda)$ is analytic in $\operatorname{Im}\lambda > 0$ and is continuous in $\operatorname{Im}\lambda \ge 0$. It should be noted that the operator, generated by the equation (1) in the $L_2(-\infty,\infty)$, has a continuous spectrum, filling the entire axis. Hence, it follows from the last equality that the function $a(\lambda)$ has no zeros in the closed upper half-plane.

Further, according to (4), (6), (7), (11) the function $a(\lambda), b(\lambda)$ have the asymptotic behavior at $\lambda \to \infty$

$$a(\lambda) = \frac{1}{2} + O\left(\lambda^{-\frac{1}{2}}\right). \quad (11)$$

A set of values $\{a(\lambda), b(\lambda), \lambda \in R\}$ is called the scattering data for the equation (1). Functions $r(\lambda), t(\lambda)$ that are defined by formulas $r(\lambda) = \frac{\overline{a(\lambda)}}{a(\lambda)}, t(\lambda) = \frac{1}{a(\lambda)}$ are called the reflection coefficient and the transmission coefficient of equation (1), respectively.The inverse scattering problem for equation (1) is recovering the coefficient $p(x)$ by the transmission coefficient $t(\lambda)$.

The central role for constructing the solution of the inverse scattering problem is played by the so-called main equation which is a linear integral equation of Fredholm type. Denote

$$F^+(x,y) = -\frac{1}{6}\int_{-\infty}^{\infty}\left(\left|\frac{t(\lambda)}{2}\right|^2 - 1\right) f_0^+(x,\lambda) f_0^+(y,\lambda) d\lambda. \quad (12)$$

Consider identities (9). We substitute the representations (6) into it, multiply them by $6^{-1}a^{-1}(\lambda) f_0^\mp(y,\lambda)$, and integrate around the whole line. Then, applying the residue theorem and using (4)-(6), (9) we obtain the following theorem.

Theorem 2. *For each fixed* x *; the functions* $K^+(x,t)$, *defined in (6), satisfy the integral equation*

$$F^+(x,y) + K^+(x,y) + \int_x^{\pm\infty} K^+(x,t)F^+(t,y) = 0, \; y > x. \quad (13)$$

Applying research techniques available for main equations $[1]-[3]$, we can obtain the basic properties of the function $F^{+}(x,y)$. By means of these properties, it is proved that equations (13) are generated by completely continuous operators on $L_2(x,\infty)$. Using Fredholm alternative is proved unique solvability of main equation (13).

The solution of the inverse scattering problem can be constructed by the following algorithm.

Algorithm. *Let the transmission coefficient* $t(\lambda)$ *be given. Then*

1) Calculate the function $F^{+}(x,y)$ *by (12).*

2) Find $K^{+}(x,t)$ *by solving the main equations (13).*

3) Construct $p(x)=-2\dfrac{dK^{+}(x,x)}{dx}$.

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ƏLAVƏ XƏTTİ POTENSİALA MALİK OLAN ŞREDİNGER TƏNLİYİ ÜÇÜN SƏPİLMƏNİN TƏRS MƏSƏLƏSİ

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Bütün oxda əlavə xətti potensiala malik olan Sredinger tənliyinə baxılır. Çevirmə operatorlarının köməyilə sonsuzluqda müəyyən asimptotikalara malik olan həllər qurulmuşdur. Tərs məsələnin əsas tənliyi tapılmışdır. Həyəcanlanma əmsalını bərpa etmək üçün tərs məsələnin həlli alqoritmi verilmişdir.

Açar sözlər: *Şredinger tənliyi, çevirmə operatorları, səpilmənin tərs məsələsi, əsas tənlik*

ОБРАТНАЯ ЗАДАЧА РАССЕЯНИЯ ДЛЯ УРАВНЕНИЯ ШРЕДИНГЕРА С ДОПОЛНИТЕЛЬНЫМ ЛИНЕЙНЫМ ПОТЕНЦИАЛОМ

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Рассматривается уравнение Шредингера с дополнительным линейным потенциалом. С помощью операторов преобразования построены решения с определенными асимптотиками на бесконечности. Найдено основное уравнения обратной задачи. Указан алгоритм решения обратной задачи для восстановления возмущенного коэффициента.

Ключевые слова: *уравнение Шредингера, операторы преобразования, обратная задача рассеяния, основное уравнение*