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ON WEIGHTED ZORKO SUBSPACES AND RIESZ TYPE THEOREMS FOR ANALYTIC FUNCTIONS

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In this paper the weighted Morrey space is considered on the interval $[-\pi,\pi]$ *and its Zorko subspace, in which the shift operator is continuous, is defined. Some properties of the functions from this subspace are studied. Moreover, a new version of the Riesz theorem on analytic functions from Hardy class is established.*

Keywords: *Morrey space, Riesz theorem, Hardy classes*

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Introduction

The concept of Morrey space was introduced by C. Morrey [1] in 1938 in the study of qualitative properties of the solutions of elliptic type equations with BMO (Bounded Mean Oscillations) coefficients (see also [2;3]). There appeared lately a large number of research works which considered many problems of the theory of differential equations, potential theory, maximal and singular operator theory, approximation theory, etc in Morrey-type spaces (for more details see [1-9]). It should be noted that the matter of approximation in Morrey-type spaces has only **started to be studied recently** (see, e.g., [5-8]), and many problems in this field are still unsolved. This work is just dedicated to this field.

In the paper the weighted Morrey space is considered on the interval $[a,b]$ and its Zorko subspace, in which the shift operator is continuous, is defined. Some properties of the functions from this subspace are studied. Moreover, a new version of the Riesz theorem on analytic functions from Hardy class is established.

Needful Information

First define the Morrey-type spaces. Let Γ be some rectifiable Jordan curve on the complex plane C. By $|M|_\Gamma$ we denote the linear Lebesgue measure of the set $M \subset \Gamma$. All the constants throughout this paper (can be different in different places) will be denoted by c.

By Morrey-Lebesgue space $L^{p,\alpha}(\Gamma)$, $0<\alpha\le 1,\ p\ge 1$, we mean the normed space of all measurable functions $f(\cdot)$ on Γ with the finite norm

$$\|f\|_{L^{p,\alpha}(\Gamma)} = \sup_B \left(|B\cap\Gamma|_\Gamma^{\alpha-1} \int_{B\cap\Gamma} |f(\xi)|^p |d\xi| \right)^{1/p} < +\infty .$$

$L^{p,\alpha}(\Gamma)$ is a Banach space with $L^{p,1}(\Gamma)=L_p(\Gamma)$, $L^{p,0}(\Gamma)=L_\infty(\Gamma)$. Similarly we define the weighted Morrey-Lebesgue space $L_\mu^{p,\alpha}(\Gamma)$ with the weight function $\mu(\cdot)$ on Γ equipped with the norm

$$\|f\|_{L_\mu^{p,\alpha}(\Gamma)} = \|f\mu\|_{L^{p,\alpha}(\Gamma)},\ f\in L_\mu^{p,\alpha}(\Gamma).$$

The inclusion $L^{p,\alpha_1}(\Gamma)\subset L^{p,\alpha_2}(\Gamma)$ is valid for $0<\alpha_1\le\alpha_2\le 1$. Thus, $L^{p,\alpha}(\Gamma)\subset L_1(\Gamma)$, $\forall\alpha\in(0,1]$, $\forall p\ge 1$. For $\Gamma=[-\pi,\pi]$ we will use the notation $L^{p,\alpha}(-\pi,\pi)=L^{p,\alpha}$.

More details on Morrey-type spaces can be found in [4-9].

Zorko subspace $M_\rho^{p,\alpha}$

Let $\rho:[-\pi,\pi]\to(0,+\infty)$ be some weight function and consider the space $M_\rho^{p,\alpha}$. It is easy to see that if $\rho\in L^{p,\alpha}$, then

$C[-\pi,\pi]\subset M_{\rho}^{p,\alpha}$ is true. Indeed, let $f\in C[-\pi,\pi]$. Without loss of generality, we assume that the function f periodically continued on the whole axis.

We have

$$|f(x+\delta)-f(x)|\le \|f(\cdot+\delta)-f(\cdot)\|_{\infty}\to 0,\quad \delta\to 0.$$

Consequently

$$\|f(\cdot+\delta)-f(\cdot)\|_{p,\alpha;\rho}=\|(f(\cdot+\delta)-f(\cdot))\rho(\cdot)\|_{p,\alpha}\le$$
$$\le \|f(\cdot+\delta)-f(\cdot)\|_{\infty}\|\rho(\cdot)\|_{p,\alpha}\to 0,\ \delta\to 0.$$

Hence, we have $f\in M_{\rho}^{p,\alpha}$.

Let us show that the set of infinitely differentiable functions is dense in $M_{\rho}^{p,\alpha}$. Consider the following averaged function

$$\omega_{\varepsilon}(t)=\begin{cases} c_{\varepsilon}\exp\left(-\dfrac{\varepsilon^2}{\varepsilon^2-|t|^2}\right), & |t|<\varepsilon,\\ 0, & |t|\ge\varepsilon,\end{cases}$$

Where

$$c_{\varepsilon}\int_{-\infty}^{+\infty}\omega_{\varepsilon}(t)dt=1.$$

Take $\forall f\in M_{\rho}^{p,\alpha}$ and consider the convolution $f*g$:

$$(f*g)(t)=\int_{-\infty}^{+\infty}f(t-s)g(s)ds,$$

and let

$$f_{\varepsilon}(t)=(\omega_{\varepsilon}*f)(t)=(f*\omega_{\varepsilon})(t).$$

It is clear that f_{ε} is infinitely differentiable on $[-\pi,\pi]$. We have

$$\|f_{\varepsilon}-f\|_{p,\alpha;\rho}=\left\|\int_{-\infty}^{+\infty}\omega_{\varepsilon}(s)f(\cdot-s)ds-f(\cdot)\right\|_{p,\alpha;\rho}=$$
$$=\left\|\int_{-\infty}^{+\infty}\omega_{\varepsilon}(s)[f(\cdot-s)-f(\cdot)]ds\right\|_{p,\alpha;\rho}.$$

Applying Minkowski inequality (11) to this expression, we obtain

$$\|f_{\varepsilon}-f\|_{p,\alpha;\rho}\le\int_{-\infty}^{+\infty}\omega_{\varepsilon}(s)\|f(\cdot-s)-f(\cdot)\|_{p,\alpha;\rho}ds=$$
$$=\int_{-\varepsilon}^{\varepsilon}\omega_{\varepsilon}(s)\|f(\cdot-s)-f(\cdot)\|_{p,\alpha;\rho}ds\le$$
$$=\sup_{|s|\le\varepsilon}\|f(\cdot-s)-f(\cdot)\|_{p,\alpha;\rho}\to 0,\ \varepsilon\to 0.$$

The following theorem is true.

Theorem 1. Let $\rho\in L^{p,\alpha}$, $1<p<+\infty$, $0<\alpha\le 1$. *Then infinitely differentiable functions are dense in* $M_{\rho}^{p,\alpha}$.

Let $P_r(t)=\dfrac{1-r^2}{1-2r\cos t+r^2}$, $0<r<1$, be the Poisson kernel for the unit disc. Assume

$$BM_{\rho}^{p,\alpha}=\left\{f\in M_{\rho}^{p,\alpha}:\sup_{s\in R}\|f(\cdot-s)\|_{p,\alpha;\rho}<+\infty\right\}.$$

Let $f\in BM_{\rho}^{p,\alpha}\cap M_{\rho}^{p,\alpha}$. The following theorem is proved .

Theorem 2. *Let* $f\in BM_{\rho}^{p,\alpha}\cap M_{\rho}^{p,\alpha}$, $1<p<+\infty$, $0<\alpha\le 1$. *Then* $\|P_r*f-f\|_{p,\alpha;\rho}\to 0$, *as* $r\to 1-0$.

Riesz type theorems

Let $F\in H_{+}^{p,\alpha}$, $1<p<+\infty$, $0<\alpha\le 1$. Then $F(re^{i\theta})\to F^{+}(e^{i\theta})$, a.e. $\theta\in[-\pi,\pi]$, as $r\to 1-0$. Consequently

$$|F(re^{i\theta})|^p\to|F^{+}(e^{i\theta})|^p,\ r\to 1-0,$$
$$\text{a.e. } \theta\in[-\pi,\pi].$$

Let $\{r_n\}_{n\in N}\subset(0,1)$: $r_n\to 1$, $n\to\infty$, be arbitrary sequence and $E_0\subset[-\pi,\pi]$ – be an arbitrary measurable set. Take an arbitrary number $\delta > 0$. Then, by Egorov's theorem $\exists E \subset [-\pi, \pi]$, such that $|e| < \delta$, where $e = [-\pi, \pi] \setminus E$ ($|e|$ − is a Lebesgue measure of the set e) and the sequence $F(r_n e^{i\theta})$ uniformly converges to $F^+(r_n e^{i\theta})$ as $n \to \infty$ on E . Consequently

$$\lim_{n\to\infty} \int_E |F(r_n e^{i\theta})|^p d\theta = \int_E |F^+(e^{i\theta})|^p d\theta .$$

As it follows from Riesz theorem

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |F(r_n e^{i\theta})|^p d\theta = \int_{-\pi}^{\pi} |F^+(e^{i\theta})|^p d\theta ,$$

holds. From these two relations we directly obtain

$$\lim_{n\to\infty} \int_e |F(r_n e^{i\theta})|^p d\theta = \int_e |F^+(e^{i\theta})|^p d\theta . \quad (1)$$

Now, take $\forall \varepsilon > 0$. From the absolute continuity of the integral, it follows that $\exists \delta$, such that for $|e| < \delta$ the following inequality holds.

$$\int_e |F^+(e^{i\theta})|^p d\theta < \frac{\varepsilon}{2}.$$

Then from the relation (1) follows that $\exists n_1$:

$$\int_e |F(r_n e^{i\theta})|^p d\theta < \varepsilon , \ \forall n \ge n_1 .$$

It is clear that

$$\lim_{n\to\infty} \int_{E_0 \cap E} |F(r_n e^{i\theta})|^p d\theta = \int_{E_0 \cap E} |F^+(e^{i\theta})|^p d\theta .$$

Consequently, $\exists n_2 \in N$:

$$\left| \int_{E_0 \cap E} |F(r_n e^{i\theta})|^p d\theta - \int_{E_0 \cap E} |F^+(e^{i\theta})|^p d\theta \right| < \varepsilon ,$$
$$\forall n \ge n_2 .$$

We have

$$\left| \int_{E_0} |F(r_n e^{i\theta})|^p d\theta - \int_{E_0} |F^+(e^{i\theta})|^p d\theta \right| \le$$
$$\le \left| \int_{E_0 \cap E} |F(r_n e^{i\theta})|^p d\theta - \int_{E_0 \cap E} |F^+(e^{i\theta})|^p d\theta \right| +$$
$$+ \int_{E_0 \cap e} |F(r_n e^{i\theta})|^p d\theta + \int_{E_0 \cap e} |F^+(e^{i\theta})|^p d\theta \le$$
$$< \varepsilon + \int_e |F(r_n e^{i\theta})|^p d\theta + \int_e |F^+(e^{i\theta})|^p d\theta < 3\varepsilon ,$$
$$\forall n \ge n_0 ,$$

where $n_0 = \max(n_1; n_2)$. From the arbitrariness of $\varepsilon > 0$ hence it immediately follows

$$\lim_{n\to\infty} \int_{E_0} |F(r_n e^{i\theta})|^p d\theta = \int_{E_0} |F^+(e^{i\theta})|^p d\theta .$$

Since, $\{r_n\}_{n\in N}$ is an arbitrary sequence, it is clear that

$$\lim_{r\to\infty} \int_{E_0} |F(r_n e^{i\theta})|^p d\theta = \int_{E_0} |F^+(e^{i\theta})|^p d\theta .$$

On the other hand have

$$\left(\int_{E_0} |F(r_n e^{i\theta})^p - F^+(e^{i\theta})|^p d\theta \right)^{1/p} \le$$
$$\le \left(\int_{E_0 \cap E} |F(r_n e^{i\theta}) - F^+(e^{i\theta})|^p d\theta \right)^{1/p} +$$
$$+ \left(\int_{E_0 \cap e} |F(r_n e^{i\theta})|^p d\theta \right)^{1/p} + \left(\int_{E_0 \cap e} |F^+(e^{i\theta})|^p d\theta \right)^{1/p} .$$

Hence, considering the above reasoning we obtain

$$\lim_{r\to 1-0} \int_{E_0} |F(re^{i\theta}) - F^+(e^{i\theta})|^p d\theta = 0 .$$

Thus, the following analogue of Riesz theorem is true.

Theorem 3. *Let* $F \in H_p^+$, $1 < p < +\infty$, *and* $E_0 \subset [-\pi, \pi]$ − *be an arbitrary measurable set. Then the following relations are true*

$$\lim_{r\to 1-0}\int_{E_0}\left|F\left(re^{i\theta}\right)\right|^p d\theta=\int_{E_0}\left|F^+\left(e^{i\theta}\right)\right|^p d\theta,$$

$$\lim_{r\to 1-0}\int_{E_0}\left|F\left(re^{i\theta}\right)-F^+\left(e^{i\theta}\right)\right|^p d\theta=0.$$

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ÇƏKİLİ ZORKO ALTFƏZALARI VƏ ANALİTİK FUNKSİYALAR ÜÇÜN RİSS TİP TEOREMLƏR

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İşdə $[-\pi,\pi]$ parçasında çəkili Morri fəzasına baxılmış və onun sıçrayış operatorunun kəsilməz olduğu Zorko altfəzaları təyin olunmuşdur. Bu altfəzadan olan funksiyaların bəzi xassələri öyrənilmişdir. Bundan əlavə Hardi sinfindən olan analitik funksiyalar haqqında Riss teoreminin yeni isbatı verilmişdir.

Açar sözlər: *Morri fəzası, Riss teoremi, Hardi sinifləri*

О ВЕСОВЫХ ПОДПРОСТРАНСТВАХ ЗОРКО И ТЕОРЕМАХ ТИПА РИССА ДЛЯ АНАЛИТИЧЕСКИХ ФУНКЦИЙ

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В работе рассматривается весовое пространство Морри на отрезке $[-\pi,\pi]$ и определяются его Зорко подпространства, в котором оператор сдвига непрерывен. Изучаются некоторые свойства функций этого подпространства. Кроме того, приводится новый вариант доказательства теоремы Рисса об аналитических функциях из класса Харди.

Ключевые слова: *пространство Морри, теорема Рисса, классы Харди*