SPIN ROTATION UNDER LORENTZ TRANSFORMATIONS AND ITS GEOMETRIC INTERPRETATIONS

B. A. Rajabov [a*]

[a] *N. Tusi Shamakhy Astrophysical Observatory,*
Azerbaijan National Academy of Sciences, Shamakhy region, Azerbaijan

This review presents the main points of the theory of Wigner's representations of the quantum-mechanical Poincarè group $ISO(3,1)$ and its application in the theory of elementary particles. Explicit formulas for the angles of rotation of the spin during Lorentz turns, as well as geometric interpretations of the results of Wigner's theory in terms of spherical and hyperbolic geometries are given. The results admit a direct generalization to cosmological groups $SO(4,1)$ and $SO(3,2)$.

Keywords: Poincarè group $ISO(3,1)$, spherical geometry, hyperbolic geometry, unitary ray representations, small group, Wigner's operator

1. INTRODUCTION

Groups of space-time movements play a very important role in the theory of elementary particles and cosmology. The concept of invariance groups establishes a link between high energy physics and cosmology.

In modern cosmology and the physics of elementary particles, the space-time of Minkowski and the worlds of de Sitter are often used, especially.

The special role of these cosmological models is related to the fact that they possess maximal groups of motions: the Poincarè group $P_+^\uparrow \equiv ISO(3,1)$ in the case of the Minkowski world and the groups $SO(4,1)$ and $SO(3,2)$ in the case of the de Sitter worlds.

The theory of elementary particles is a relativistic theory. The foundations of this theory are the quantum-mechanical principles of nature description and the Einstein relativity principle [1,2].

* E-mail: balaali.rajabov@mail.ru

In this article, we will limit ourselves to the consideration the Minkowski and the Poincarè group.

The main result of the synthesis of the principles of the quantum-mechanical description of the nature and relativity of Einstein is that the role of the group of space-time symmetries is played not by the Poincarè group $P_+^\uparrow$ but its universal covering group, the so-called quantum-mechanical Poincarè group $\overline{P_+^\uparrow}$, [3]. A namely, Unitary Irreducible Representations (UIR) of the quantum-mechanical Poincarè group $\overline{P_+^\uparrow}$ describe the state of elementary particles. Invariants of the Poincarè group $\overline{P_+^\uparrow}$ (mass and spin) characterize the invariant properties of elementary particles, and the generators of the Poincarè group determine the set of observables that determine the states of elementary particles during independent measurements.

The relativistic dynamics of a free particle is uniquely determined by the behavior of its state vector during the transformation $\bar{g} \in \overline{P_+^\uparrow}$:

$$|\gamma_g >= U(\bar{g})|\gamma > .$$

Finding the explicit form of the operator $U(\bar{g})$ acting on the state vector is equivalent to solving the equation of motion for a free particle. $U(\bar{g})$ operators for an elementary particle constitute representations of the Poincarè group $\overline{P_+^\uparrow}$.

2. QUANTUM MECHANICAL GROUP POINCARÈ

An element of the classical Poincarè group $g = (a, \Lambda)$ consists of 4-vector a^μ and 4×4 Lorentz matrices $\Lambda_{\mu\nu} \in SO(3,1)$, $\mu, \nu = 0, 1, 2, 3$. In the quantum-mechanical Poincarè group Lorentz transformations are given by unimodular 2×2-matrices $A \in SL(2, C)$, i.e. $\det A = 1$, and the translation vector a^μ Hermitian matrices are matched $a = \sigma_\mu a^\mu$. Thus, the element of the group $\bar{g} \in \overline{P_+^\uparrow}$ is written as: $\bar{g} = (a, A)$.

Here, σ_μ - Pauli matrices [2], supplemented by the unit matrix:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For Pauli matrices, well-known commutation and anti-commutation relations are satisfied[1)]:

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k, \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}\sigma_0, \qquad i, j, k = 1, 2, 3,$$

[1)] By double-repeated indices, the Einstein summation rule is implied.

where δ_{ij} is Kronecker symbol, and ε_{ijk} is the symbol of Levi-Civita[2)]

Square brackets mean a commutator, curly – anti-commutator.

In addition, the Pauli matrices are orthogonal in the following sense:

$$\frac{1}{2}\mathrm{Sp}\left(\sigma_\mu \sigma_\nu\right) = \delta_{\mu\nu}$$

The correspondence between Lorentz transformations $SO(3,1)$ and $SL(2,C)$ transformations is established as follows. Transformation of the matrix $b = \sigma_\mu b^\mu$:

$$b \longrightarrow b' = AbA^+$$

corresponds to the Lorentz transformation of the following form:

$$b^\mu \longrightarrow b'^\mu = \Lambda^\mu_{.\nu} b^\nu,$$

and takes place:

$$\begin{aligned} \Lambda(A) &= \Lambda(-A); \\ \Lambda^\mu_{.\nu}(A) &= \frac{1}{2}\mathrm{Sp}\left(\sigma_\mu A \sigma_\nu A^+\right). \end{aligned} \tag{1}$$

It is easy to see that the matrix:

$$\begin{aligned} A_L &= \sigma_0 \cosh\frac{\omega}{2} + \left(\sigma_k n^k\right)\sinh\frac{\omega}{2} = \\ &= \exp\left[\left(\sigma_k n^k\right)\frac{\omega}{2}\right], \qquad A_L^+ = A_L, \end{aligned} \tag{2}$$

describes the pure Lorentz transformation in the direction $\mathbf{n}$ with speed $v = \tanh\omega$ without any rotations. And the matrix:

$$\begin{aligned} A_R &= \sigma_0 \cos\frac{\omega}{2} + i\left(\sigma_k n^k\right)\sin\frac{\omega}{2} = \\ &= \exp\left[i\left(\sigma_k n^k\right)\frac{\omega}{2}\right], \qquad A_R^+ = A_R^{-1},, \end{aligned} \tag{3}$$

describes rotation by the angle ω around the unit vector $\mathbf{n}$.

The multiplication rule for elements of the classical Poincarè group $P_+^\uparrow$ has a form:

$$\left(a_1, \Lambda_1\right)\left(a_2, \Lambda_2\right) = \left(a_1 + \Lambda_1 a_2, \Lambda_1\Lambda_2\right),$$

and for the elements of the quantum-mechanical Poincarè group $\overline{P_+^\uparrow}$, the multiplication law is as follows:

$$\left(a_1, A_1\right)\left(a_2, A_2\right) = \left(a_1 + A_1 a_2 A_1^+, A_1 A_2\right). \tag{4}$$

[2)] Hereinafter, the Latin indexes run through the values 1,2,3 and the Greek indexes run through the values 0,1,2,3.

In addition, the element $(0, I)$ plays the role of the unit, and the inverse element has the form:

$$(a, A)^{-1} = \left(-A^{-1} a A^{-1^{+}}, A^{-1}\right).$$

The inhomogeneous Lorentz group, i.e. the Poincarè group is a 10-parameter continuous group whose generators consist of operators iP_λ and $iM_{\mu\nu} = -iM_{\nu\mu}$.

Here

iP_λ is the translation generator in x^λ direction;

$iM_{\mu\nu}$ is the rotation generator in the $x^\mu x^\nu$ plane.

For these generators, the following commutation relations are valid:

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \\ [M_{\mu\nu}, M_{\varrho\sigma}] &= i\left(g_{\mu\sigma} M_{\nu\varrho} + g_{\nu\varrho} M_{\mu\sigma} - g_{\mu\varrho} M_{\nu\sigma} - g_{\nu\sigma} M_{\mu\rho}\right), \\ [M_{\mu\nu}, P_\sigma] &= i\left(g_{\nu\sigma} P_\mu - g_{\mu\sigma} P_\nu\right). \end{aligned} \tag{5}$$

These generators correspond to 10 one-parameter transformation subgroups of the Poincarè group and form the main observables in quantum mechanics.

Here,

P_μ is called an energy-momentum or 4-momentum;

and 3-vector

$$M_m = \frac{1}{2}\varepsilon_{mnk} M^{nk} = \left(M_{23}, M_{31}, M_{12}\right),$$

is called an angular momentum.

As a rule, instead of the M_{0j} components, a 3-vector is introduced:

$$\mathbf{N} = \left(M_{01}, M_{02}, M_{03}\right).$$

Then the commutation relations (5) for the components of $M_{\mu\nu}$ take the following form:

$$\begin{aligned} [M_i, M_j] &= i\varepsilon_{ijl} M_l, \\ [M_i, N_j] &= i\varepsilon_{ijl} N_l, \\ [N_i, N_j] &= -i\varepsilon_{ijl} M_l; \quad i, j, l = 1, 2, 3. \end{aligned} \tag{6}$$

The solution of the problem of finding all UIR's of the Poincarè group is equivalent to the problem of constructing representations of the commutation relations (5) using self-adjoint operators. First of all, it is necessary to find invariants groups, i.e. build using generators $M_{\mu\nu}$ and P_λ quantities commuting with all generators. Operators that do not change during transformations and for each irreducible representation proportional to one operator are called Casimir operators. To find Casimir operators of the Poincaré group we introduce the polarization vector (Pauli-Lubansky vector):

$$W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma} M^{\nu\lambda} P^\sigma. \tag{7}$$

This vector has the following properties:

$$W_\mu P^\mu = 0, \qquad [W_\lambda, P_\nu] = 0.$$

For the vector W^μ, the following commutation relations hold:

$$\begin{aligned} [M_{\mu\nu}, W_\rho] &= i\left(g_{\mu\rho} W_\nu - g_{\nu\varrho} W_\mu\right), \\ [W_\mu, W_\nu] &= i\varepsilon_{\mu\nu\varrho\sigma} W^\varrho P^\sigma. \end{aligned} \tag{8}$$

It is easy to check that the scalar operators:

$$m^2 = P^\mu P_\mu,$$

$$w^2 = W^\mu W_\mu = M_{\mu\sigma} M^{\nu\sigma} P^\mu P_\nu - \frac{1}{2} M_{\mu\nu} M^{\mu\nu} P_\sigma P^\sigma$$

commute with all generators $M_{\mu\nu}$ and P_λ. Therefore, for all irreducible representations of the Poincarè group, they are proportional to a unit operator and their own values can be distinguished equivalence classes of irreducible representations.

We will not be interested in the case of $m^2 < 0$, since in this case the particle energy can be given arbitrarily large in absolute values using the Lorentz transformations, i.e. the energy spectrum from below is unbounded (it is not a real physical case!). It should be noted that for representations corresponding to real physical particles, the energy is always nonnegative $P_0 \geq 0$. On the other hand, non-relativistic limiting cases are possible for such representations, whereas representations with $m^2 < 0$ do not possess these properties.

Thus, we will restrict ourselves only to the case of $m^2 \geq 0$. In this case of one can enter a spin vector:

$$J^k = -\frac{1}{m} W^\lambda n_\lambda^{(k)}, \tag{9}$$

where $n_\lambda^{(k)}$ is the unit 4-vector orthogonal to 4-vector momentum P^μ:

$$n_\lambda^{(k)} n^{\lambda(k')} = -\delta_{kk'}; \qquad n_\lambda^{(k)} P^\lambda = 0.$$

The spin vector has the following properties:

$$\begin{aligned} \mathbf{J}^{\,2} &= \frac{1}{m^2} \overrightarrow{W}^2 = J(J+1); \\ \left[J^i, J^j\right] &= i\varepsilon_{ijk} J^k. \end{aligned} \tag{10}$$

Unitary irreducible representations (UIR) of the Poincarè group are characterized by the values of the spin J, the mass m and the energy sign (for $m^2 > 0$). For physical states, energy is always positive.

To study the unitary representations of the inhomogeneous Lorentz group, it is advisable to define a basis in the space of representations. The physical state vectors of irreducible representations may be characterized by the eigenvalues of the operators included in the full algebra of observables. These operators with necessity will be functions of generators $M_{\mu\nu}$ and P_λ. One of these bases is the *canonical basis*, which is possible for $m^2 > 0$.

In the canonical basis, the components of the momentum p_j and spin projection σ in the direction perpendicular to the 4-momentum vector P^μ are diagonalized. In other words, a full set of mutually commuting operators will include the following operators:

$$J_3 = \frac{1}{m} W^\lambda n_\lambda^{(3)} = \sigma; \quad m^2, J^2, p_j,$$

where $\left(n^{(3)}\right)^2 = -1$, $P^\lambda n_\lambda^{(3)} = 0$, and the state vector in the Dirac notation, [1] is written as: $|\mathbf{p}, \sigma; m, J >$.

It should be noted that although this set we chose among the complete set of commuting operators, it is not a complete set of commuting physical quantities. In general, there are other operators (for example, operators of lepton or baryon charge), commuting operators of a group. Eigenvalues of these operators together with $\mathbf{p}$ and σ characterize the states of physical systems. Therefore, state vectors in the space of irreducible representations in a more general form will be written as follows:

$$|\mathbf{p}, \sigma; m, J; \zeta >, \tag{11}$$

where ζ is the complementary set of physical quantities (p_j, J_3) to full set of observed other variables. Very often for brevity, we omit the characters m, J, ζ in (11) and denote a vector of canonical basis in the form: $|\mathbf{p}, \sigma >$.

Our aim is to find the explicit form of a unitary transformation of the state vector:

$$|\mathbf{p}, \sigma > \longrightarrow U(\bar{g})|\mathbf{p}, \sigma >, \ \bar{g} \in \overline{P_+^\uparrow}.$$

For inhomogeneous transformations from (4) we get:

$$(a, A) = (a, \hat{1})(0, A) = (0, A)\left(A^{-1} a A^{-1^+}, \hat{1}\right). \tag{12}$$

Hereinafter, $\hat{1}$ is a unit element (or a matrix, depending on from context).

Therefore, to find unitary representations of $U(a, A)$, it suffices to find representations of the subgroup of translations $U(a, 1)$ and the subgroups of homogeneous transformations $U(0, A)$.

UIR's of the quantum-mechanical Poincarè group $\overline{P_+^\uparrow}$ is found by Wigner [3]. The method of induced representations [4] is used to build representations of $U(a, A)$ based on the concept of stationary groups.

3. STATIONARY GROUP AND WIGNER OPERATOR

In this section, we study the translation subgroup. Representation of this subgroup satisfies the commutative law

$$U(a_1, \hat{1})U(a_2, \hat{1}) = U(a_1 + a_2, \hat{1}), \quad U(0, \hat{1}) = \hat{1}.$$

The unitary representations of this 4-parameter Abelian subgroup are expressed by the exponent. The translation to the 4-vector a^μ corresponds to the exponent

$$U(a, \hat{1}) = e^{iP_\mu a^\mu}.$$

In states that diagonalizing the momentum (momentum representation), this operator acts as follows

$$U(a, \hat{1})|\mathbf{p}, \sigma >= e^{ipa}|\mathbf{p}, \sigma > . \tag{13}$$

For given m and J, the completeness condition for state vectors has a form

$$\sum_\sigma \int d\mu(\mathbf{p})|\mathbf{p}, \sigma >< \mathbf{p}, \sigma| = 1.$$

Here $d\mu(\mathbf{p})$ is an invariant measure in the space of momenta. In the mass shell $p^2 = m^2$, the invariant volume takes the form

$$d\mu(\mathbf{p}) = \delta(p^2 - m^2)d^4p = \frac{d^3p}{2p_0}, \quad p_0 > 0.$$

Then the condition of orthonormality for the canonical basis takes the following form

$$< \mathbf{p}', \sigma'|\mathbf{p}, \sigma >= 2p_0\delta(\mathbf{p} - \mathbf{p}')\delta_{\sigma\sigma'}, \quad p_0 > 0. \tag{14}$$

The transformation law of the 4-momentum matrix $p = P^\mu \sigma_\mu$ is as follows

$$\begin{aligned} p' &= ApA^+, \\ p'^\mu &= \Lambda^\mu_{\cdot\nu}(A)p^\nu. \end{aligned} \tag{15}$$

Consider a set of the matrices $\tilde{A}(p)$ that preserve the invariant momentum matrix p

$$\tilde{A}(p)p\tilde{A}^+(p) = p, \quad \tilde{A}(p) \subset SL(2, C). \tag{16}$$

It is obvious that

- the matrices $\tilde{A}(p)$ form a subgroup of the group of unimodular 2×2-matrices over the field of complex numbers. This subgroup is called the stationary subgroup of momentum p and is denoted by $L(p)$;
- the stationary subgroups for different values of the momentum p are isomorphic if these values of the momentums can be translated into others Lorentz transformations.

Choose the element $\alpha(p,\mathring{p}) \in SL(2,C)$ which converts the momentum $\mathring{p}$ into p

$$p = \alpha(p,\mathring{p})\mathring{p}\alpha^{+}(p,\mathring{p}). \tag{17}$$

Then the elements $\tilde{A}(p)$ and $\tilde{A}(\mathring{p})$ of the stationary subgroups $L(p)$ and $L(\mathring{p})$ correspondingly can be related as follows:

$$\tilde{A}(\mathring{p}) = \alpha^{-1}(p,\mathring{p})\tilde{A}(p)\alpha(p,\mathring{p}). \tag{18}$$

Indeed, from (15)-(17) we have:

$$\tilde{A}(\mathring{p})\mathring{p}\tilde{A}^{+}(\mathring{p}) = \mathring{p}.$$

Thus, it is established that all stationary subgroups of $L(p)$ are isomorphic to the stationary subgroup of $L(\mathring{p})$ fixed momentum $\mathring{p}$. In this construction, $\mathring{p}$ is called the standard momentum, and $\alpha(p,\mathring{p})$ is called the Wigner operator. In the future we will use the abbreviated designation

$$\alpha(p) \equiv \alpha(p,\mathring{p}).$$

Let be

$$\alpha(p)\mathring{p}\alpha^{+}(p) = p, \tag{19}$$
$$\alpha(p')\mathring{p}\alpha^{+}(p') = p'. \tag{20}$$

Then from (16) and (18) - (20) we have

$$\alpha^{-1}(p')A\alpha(p) = \tilde{A}(\mathring{p},A). \tag{21}$$

Thus, we have established that any element of $A \in SL(2,C)$ can be represented as

$$A = \alpha(p')\tilde{A}(\mathring{p},A)\alpha^{-1}(p). \tag{22}$$

In formulas (21)-(22) we explicitly distinguished the dependence of the choice of an element of the stationary subgroup $L(\mathring{p})$ on A.

Equation (22) plays the main role in the construction of representations of the Poincaré group using representations of a stationary subgroup.

4. CANONICAL BASIS AND WIGNER OPERATOR UNDER $m^2 > 0$

For this case, choose the standard momentum $\mathring{p}^\mu = (m, 0, 0, 0)$. As the standard momentum matrix take $\mathring{p} = m\hat{1}$. Then from (16) we have

$$\tilde{A}(\mathring{p})\tilde{A}^+(\mathring{p}) = \hat{1},$$

i.e. for time-like momentums the matrix $\tilde{A}(\mathring{p})$ is unitary: $\tilde{A}(\mathring{p}) \in SU(2)$. The group $SU(2)$ and its representations, as a universal covering group of 3-dimensional rotations, are well studied [5].

From the definition, it is obvious that the Wigner operator is determined with an accuracy to the element of the stationary subgroup, in this case, the rotations from groups $SU(2)$. Using this arbitrariness we define the operator Wigner so that he translated the state of rest to a state motions by a purely Lorentz transformation without any spatial rotation. From (2) it can be seen that this is equivalent to the hermiticity Wigner operator:

$$\alpha(p) = \alpha^+(p).$$

Then the solution of equation (17) takes the following form

$$\alpha(p)\mathring{p}\alpha^+(p) = m\alpha^2(p) = p.$$

Finally, we have

$$\alpha(p) = \frac{m + p_0 + \left(\sigma_k p^k\right)}{\sqrt{2m(m + p_0)}}. \tag{23}$$

Thus, we obtained an explicit form of the Wigner operator in the canonical basis.

5. SPIRAL BASIS AND WIGNER OPERATOR UNDER $m^2 > 0$

In some problems it is advisable to use a spiral basis. The spiral basis differs from the canonical basis in that the full set of observables instead of $J_3 = \frac{1}{m}W^\lambda n_\lambda^{(3)}$ is included λ helicity

$$\lambda = -\frac{W^0}{|\mathbf{p}\,|} = \frac{(\mathbf{Mp})}{|\mathbf{p}\,|}. \tag{24}$$

Helicity is the projection of the total angular momentum in the direction of the motion. Obviously, the helicity, an invariant quantity with respect to space rotations and translations. The eigenvalues of the helicity operator are exactly the same as for the spin projection:

$$\lambda = -J, -(J-1), \ldots, J-1, J.$$

We define the Wigner operator in the form:

$$h(\mathbf{p}\,) \equiv h(p,\vartheta,\varphi) = R(\varphi,\vartheta,0)e^{\frac{1}{2}\sigma_3\beta}, \tag{25}$$

where

$$\sinh\beta = \frac{|\mathbf{p}\,|}{m}, \qquad 0 \le \varphi \le \pi,\ 0 \le \vartheta \le \pi.$$

Here ϑ is the angle between the momentum and the z axis, and φ is the angle between the projection of the momentum in the (x,y) plane and the x axis.

The Wigner operator $h(\mathbf{p}\,)$ in the formula (25) corresponds to the following transformations: first, using the purely Lorentz transformation $\exp\bigl(\frac{1}{2}\sigma_3\beta\bigr)$ standard momentum $\mathring{p}$ is translated into momentum, with the module $|\mathbf{p}\,|$ and directed in the positive direction of the z axis, and then this momentum by rotating

$$R(\varphi,\vartheta,0) = e^{-i\frac{1}{2}\sigma_3\varphi}e^{-i\frac{1}{2}\sigma_2\vartheta}. \tag{26}$$

combined with a given momentum $\mathbf{p}$. In this case, the momentum $|\mathbf{p}|$ is parametrized as follows

$$p^\mu = m(\cosh\beta, \sinh\beta\sin\vartheta\cos\varphi, \sinh\beta\sin\vartheta\sin\varphi, \sinh\beta\cos\vartheta). \tag{27}$$

If the momentum $\mathbf{p}$ is directed in the negative direction of the z axis, then the space rotation R must be chosen in the form $R(\varphi,\pi-\vartheta,0)$.

The 2×2 -matrix of the momentum p^μ has the following form

$$\begin{aligned} p \equiv p^\mu\sigma_\mu &= \begin{pmatrix} p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & p^0-p^3 \end{pmatrix} = \\ &= m\begin{pmatrix} \cosh\beta+\sinh\beta\cos\vartheta & e^{-i\varphi}\sinh\beta\sin\vartheta \\ e^{i\varphi}\sinh\beta\sin\vartheta & \cosh\beta-\sinh\beta\cos\vartheta \end{pmatrix} = \\ &= m\begin{pmatrix} e^{\beta}\cos^2\frac{\vartheta}{2}+e^{-\beta}\sin^2\frac{\vartheta}{2} & 2e^{-i\varphi}\sinh\beta\sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2} \\ 2e^{i\varphi}\sinh\beta\sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2} & e^{-\beta}\cos^2\frac{\vartheta}{2}+e^{\beta}\sin^2\frac{\vartheta}{2} \end{pmatrix}; \end{aligned} \tag{28}$$

$$\det p = m^2, \quad p^+ = p.$$

Inverse relationships are as follows

$$p^\mu = \frac{1}{2}\mathrm{Sp}\,(p\sigma_\mu)\,. \tag{29}$$

For the covariant components, we have, respectively

$$p_\mu = g_{\mu\nu} p^\nu , \qquad g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)\,, \tag{30}$$
$$p_\mu = \frac{1}{2}\mathrm{Sp}\,(p_\mu \tilde{\sigma}_\mu)\,, \qquad \tilde{\sigma} = (\sigma_0, -\boldsymbol{\sigma}) = (\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3)\,.$$

The orthogonality relation for the matrices σ and $\tilde{\sigma}$ is as follows

$$\frac{1}{2}\mathrm{Sp}\,(\sigma_\mu \tilde{\sigma}_\nu) = g_{\mu\nu}.$$

Thus, the state vector in the spiral basis takes the following form

$$|\mathbf{p}, \lambda; m, J; \zeta >\equiv |\mathbf{p}, \lambda > . \tag{31}$$

The conditions of orthonormality (14) of the state vectors in the spiral basis take the following form

$$< \mathbf{p}^{'}, \lambda' | \mathbf{p}, \lambda >= 2p_0 \delta(\mathbf{p} - \mathbf{p}^{'}) \delta_{\lambda\lambda'}, \quad p_0 > 0. \tag{32}$$

6. UNITARY REPRESENTATIONS UNDER $m^2 > 0$

Now we study the unitary representations $U(0, A)$ of the universal covering group $SL(2, C)$ of the Lorentz group SO(3,1) at $m^2 > 0$. This allows us using (12)-(13) construct representations of the Poincarè group. For the Lorentz transformation A, the state vector in the spiral basis $|\mathbf{p}, \lambda >$ is exposed unitary transformation

$$|\mathbf{p}, \lambda > \longrightarrow U(0, A)|\mathbf{p}, \lambda > .$$

The state vectors are normalized according to (32). As a result of the Lorentz transformation Λ, the momentum receives a new value

$$p' = \Lambda p,$$

and the state vector is a new eigenvalue

$$P_\mu U(0, A)|\mathbf{p}, \lambda >= U(0, A)\Lambda_\mu^{\cdot\nu} p_\nu |\mathbf{p}, \lambda >= p_\mu^{'} U(0, A)|\mathbf{p}, \lambda > .$$

The decomposition of the new state vector in the spiral basis takes the following form

$$U(0, A)|\mathbf{p}, \lambda >= \sum_\lambda |\mathbf{p}^{'}, \lambda' > C_{\lambda'\lambda}(p, A), \quad p' = \Lambda(A)p. \tag{33}$$

In particular, for the elements of the stationary subgroup $A \equiv \tilde{A}(p)$ of the momentum p we have: $\tilde{A}(p) p \tilde{A}^{+}(p) = p$.

Then from (33) we get

$$U(0, \tilde{A}(p))|\mathbf{p}, \lambda >= \sum_{\lambda'} |\mathbf{p}, \lambda' > C_{\lambda'\lambda}(p, \tilde{A}(p)).$$

This shows that the matrix $C(p, \tilde{A}(p))$ is a representation of the stationary subgroup $L(p)$. As we have already established in Section 3 the stationary subgroup for any momenta is isomorphic and, in the case of $m^2 > 0$, isomorphic to the group $SU(2)$.

To fix the Wigner operator, we use (25)

$$|\mathbf{p}, \lambda >= U(0, h(\mathbf{p}\,))|m, \lambda > . \tag{34}$$

Here, for the rest state $\lambda = J^3 = \sigma$. And for the definition of the matrix $C(p, \tilde{A}(p))$ we use the expression

$$C(p, \tilde{A}(\mathring{p})) = \mathcal{D}^J_{\lambda'\lambda}(\tilde{A}(\mathring{p})), \tag{35}$$

where $\mathcal{D}^J_{\lambda'\lambda}$ is unitary $(2J+1) \times (2J+1)$ - matrix forming the representations of the group $SU(2)$.

From (22) we get

$$U(0, A) = U(0, h(\mathbf{p}\,'))U(0, \tilde{A}(\mathring{p}, A)U(0, h^{-1}(\mathbf{p}\,)). \tag{36}$$

Now using (34) - (36) the last equation can be reduced to the following form

$$U(0, A)|\mathbf{p}, \lambda >= \sum_{\lambda'} |\mathbf{p}\,', \lambda' > \mathcal{D}^J_{\lambda'\lambda}(\tilde{A}(\mathring{p}, A)). \tag{37}$$

In order to obtain representations of the Poincarè group, it remains to add to the (37) translation (13)

$$U(0, A)|\mathbf{p}, \lambda >= e^{ipa} \sum_{\lambda'} |\mathbf{p}\,', \lambda' > \mathcal{D}^J_{\lambda'\lambda}(\tilde{A}(\mathring{p}, A)), \tag{38}$$

where $\tilde{A}(\mathring{p}, A)$ is determined from (22)

$$\tilde{A}(\mathring{p}, A) = h^{-1}(\mathbf{p}\,')Ah(\mathbf{p}\,). \tag{39}$$

The matrix $\tilde{A}(\mathring{p}, A)$ defines the rotation of the rest coordinate system and is called the Wigner rotation

$$\tilde{A}(\mathring{p}, A) \equiv R(\eta_1, \omega, \eta_2) = e^{-i\frac{1}{2}\sigma_3\eta_1} e^{-i\frac{1}{2}\sigma_2\omega} e^{-i\frac{1}{2}\sigma_3\eta_2}. \tag{40}$$

Here η_1, ω, η_2 are Euler angles (in this case Wigner angles!) of rotation R.

The dependences of Wigner angles η_1, ω, η_2 from A is calculated in the Appendix.

We consider 3 different cases of the matrix A since all other cases are a combination of them.

1. Let A be the rotation around the y axis

$$A = e^{-i\frac{1}{2}\sigma_2\psi}.$$

Then the rotation $R(\eta_1, \omega, \eta_2)$ of the rest coordinate system is given by the formulas /see. Appendix, eq.(67)

$$\begin{aligned} \omega &= 0; \\ \cos\eta &= \frac{\cos\psi - \cos\vartheta\cos\vartheta'}{\sin\vartheta\sin\vartheta'}. \end{aligned} \tag{41}$$

Here η is the angle of rotation around the z axis. And ϑ and ϑ' are the angles formed by the momentums $\mathbf{p}$ and $\mathbf{p}'$ with the axis z, respectively.

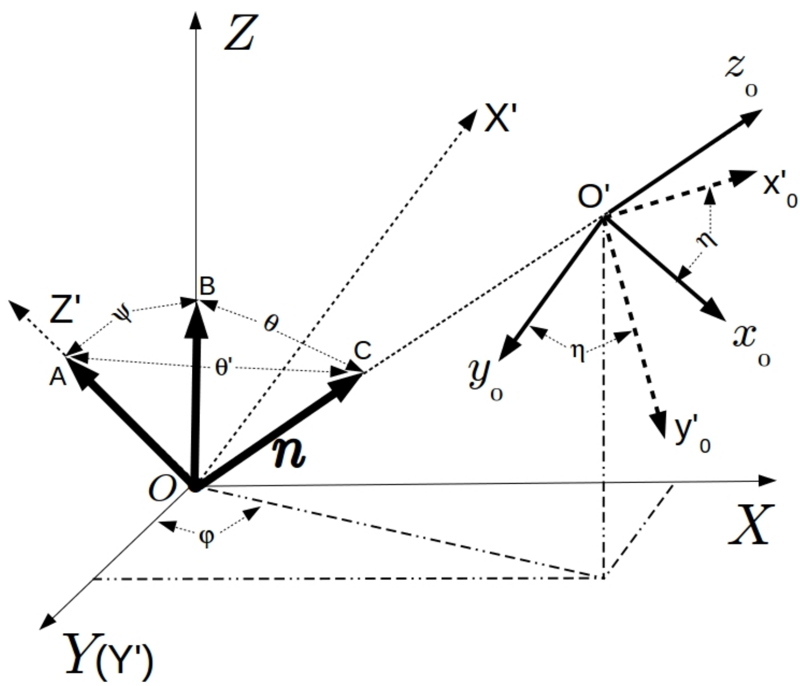

Fig. 1

The expression(41) has a simple geometric interpretation (see Fig.1). Here the vectors are oriented as follows:

$$\begin{aligned} \mathbf{n} &= \frac{\mathbf{p}}{|\mathbf{p}|} \\ \mathbf{y}_0 &\perp \mathbf{z}\,\mathbf{z}_0 \\ \mathbf{y}_0^{'} &\perp \mathbf{z}^{'}\mathbf{z}_0 \end{aligned}$$

As can be seen from Fig. 1, the spherical angle BCA is the angle between the planes (z, z_0) and (z', z_0). Since the y_0, the axis is perpendicular to these planes, the angle between them is equal to the rotation angle η of the y_0 axis around the z_0 axis. We can calculate this angle using the cosine theorem of spherical trigonometry [6–9] to the spherical triangle ABC

$$\cos C \equiv \cos\psi = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos\eta.$$

It is easy to see that this expression matches (41).

2. Let A be a rotation around the axis z

$$A = e^{-i\frac{1}{2}\sigma_3\psi}.$$

In this case, the Wigner angles are zero /see Appendix, eq.(69)

$$\eta_1 = \eta_2 = \omega = 0.$$

Indeed, since when rotating around the z-axis, the plane (z, z_0) does not change its position, the rest coordinate system will not experience any rotation (see Fig.1).

3. Let A represents the Lorentz transformation in the positive direction of the z-axis:

$$A = e^{\frac{1}{2}\sigma_3\alpha}.$$

In this case, the Wigner angles (η_1, ω, η_2) are given by the following formulas /see Appendix, eq.(72)/:

$$\begin{aligned} &\eta_1 = \eta_2 = 0. \\ &\cos\omega = \frac{\cosh\beta\cosh\beta' - \cosh\alpha}{\sinh\beta\sinh\beta'}. \end{aligned} \tag{42}$$

Here

$$\cosh\beta = \frac{p_0}{m}, \qquad \sinh\beta = \frac{|\mathbf{p}|}{m};$$
$$\cosh\beta' = \frac{p_0^{'}}{m}, \qquad \sinh\beta' = \frac{\left|\mathbf{p}^{'}\right|}{m}.$$

Indeed, as can be seen from fig.2, in this case, the (z, z_0) plane, as well as the y_0-axis, do not change their position, which means that the rest system rotates around the y_0-axis.

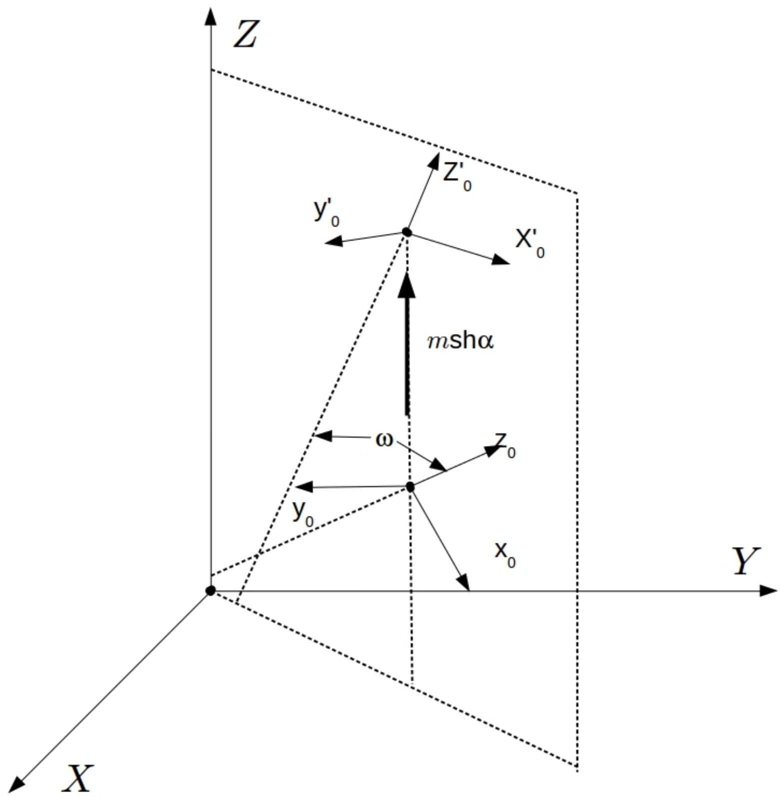

Fig. 2:

It can be shown that the expression (42) has an elegant geometric interpretation.

We introduce 4-speeds:

$$u_\mu = \frac{p_\mu}{m} = \left(\frac{p_0}{m}, -\frac{\mathbf{p}}{m}\right),$$
$$u_\mu u^\mu = \frac{p_\mu p^\mu}{m^2} = 1$$

The last equality means that in the 4-velocity space the ends of the 4-vectors lie on the surface of the hyperboloid (the point $(1,0,0,0)$ is the vertex of the hyperboloid).

Obviously, the end of the vector $\overset{\circ}{u}$ will be at the vertex of the hyperboloid. Consider a triangle formed by vertices at the points $\overset{\circ}{u}, u, u'$ (Fig. 3). The

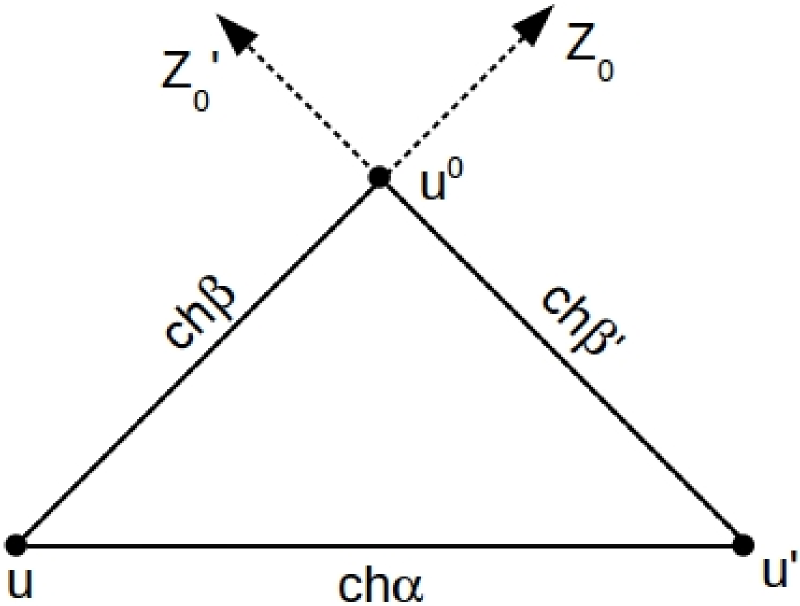

Fig. 3:

length of the sides of this triangle will be equal to the distance between its vertices:

$$u^{\mu}\mathring{u}_{\mu} = \mathring{u} = \cosh\beta,$$
$$u'^{\mu}\mathring{u}_{\mu} = \cosh\beta',$$
$$u^{\mu}u^{'}_{\mu} \equiv \cosh\alpha.$$

If applying the cosine theorem of hyperbolic geometry [6–9] for this triangle, we getting

$$\cosh\alpha \equiv u^{\mu}u^{'}_{\mu} = \cosh\beta\cosh\beta' - \sinh\beta\sinh\beta'(\mathbf{n}\mathbf{n}^{'}).$$

On the other hand, the angle between $\mathbf{n}^{'}$ and $\mathbf{n}$ is the angle of rotation around the axis y_0

$$(\mathbf{n}\,\mathbf{n}^{'}) = \cos\omega.$$

Then we get

$$\cosh\alpha = \cosh\beta\cosh\beta' - \sinh\beta\sinh\beta'\cos\omega.$$

It is easy to see that the last expression coincides with formula (72).

7. WIGNER SMALL GROUP AND OPERATOR UNDER $m^2 = 0$

For a zero mass, we choose as the standard momentum: $\overset{\circ}{p}{}^{\mu} = (1, 0, 0, 1)$. Then the expression for the 2×2 -matrix $\overset{\circ}{p}$ of the standard momentum through the Pauli matrix will look like:

$$\overset{\circ}{p} = 1 + \sigma_3. \tag{43}$$

The Wigner operator can be defined by the following formula

$$\alpha(\mathbf{p}) = \frac{1}{\sqrt{p^0 + p^3}} \begin{pmatrix} p^0 + p^3 & 0 \\ p^1 + ip^2 & 1 \end{pmatrix}. \tag{44}$$

Directly, by calculation one can verify that

$$\alpha(\mathbf{p})\overset{\circ}{p}\alpha^+(\mathbf{p}) = \alpha(\mathbf{p})(1 + \sigma_3)\alpha^+(\mathbf{p}) = |\mathbf{p}|\sigma_0 + \sigma_k p^k \equiv p. \tag{45}$$

In applications, it is sometimes convenient to define the Wigner operator as a product of a purely Lorentz transformation $L(|\mathbf{p}|)$ along the Z-axis and 3-dimensional rotation $R(\mathbf{p})$ from the Z-axis to the direction of the $\mathbf{p}$

$$\alpha(\mathbf{p}) = R(\mathbf{p})L(|\mathbf{p}|). \tag{46}$$

The rotation matrix $R(\mathbf{p})$ has the form

$$R(\mathbf{p}) = \exp\left\{ i\frac{\beta}{2}\left(\sigma_k n^k\right)\right\}, \tag{47}$$

where

$$\cos\beta = \frac{|p^3|}{|\mathbf{p}|}, \qquad \mathbf{n} = -\frac{[\mathbf{p}\,\mathbf{e}_3]}{|[\mathbf{p}\,\mathbf{e}_3]|}.$$

Here $\mathbf{e}_3$ is a unit vector along the Z-axis.

And the Lorentz transformation $L(|\mathbf{p}|)$ has the form

$$L(|\mathbf{p}|) = L^+(|\mathbf{p}|) = \frac{1}{2\sqrt{|\mathbf{p}|}}\left[|\mathbf{p}| + 1 + \sigma_3(|\mathbf{p}| - 1)\right]. \tag{48}$$

The validity of the formulas (45)-(48) can be checked by direct calculation.

For a light-like momentum, the $p^2 = 0$ stationary group is the group $ISO(2)$ - the group of translations and rotations of the Euclidean plane $E(2)$. The $ISO(2)$ group is a semi-direct product of the 2-dimensional translation group T_2 with the group $SO(2)$

$$ISO(2) = T_2 \lhd SO(2).$$

Representations of the $ISO(2)$ group are studied in [5]. It is shown that the unitary representations of this group are divided into two types:

1. Irreducible unitary representations, which are given by a nonzero real number: the restriction of this representation to the subgroup $SO(2)$ contains all the one-dimensional representations of this subgroup, consisting of exponentials. These representations are infinite-dimensional, as one would expect since the $ISO(2)$ group is a non-compact group.

2. A reduced representation in which the translation subgroup is represented by unit operators. This representation is completely reducible and decomposes into a direct sum of one-dimensional representations in which the rotation operators from the subgroup $SO(2)$ are represented as exponents.

To determine the physical meaning of these representations, it suffices to refer to the formula (7) for the polarization operator W_μ. For states with a standard momentum $\overset{\circ}{p}$, w^μ this operator takes the form

$$W^\mu\left(\overset{\circ}{p}\right) = \left(-M_{12}, E_2, -E_1, -M_{12}\right), \tag{49}$$

where

$$\begin{aligned} E_1 &= M_{01} + M_{31}, \\ E_2 &= M_{02} + M_{32}. \end{aligned} \tag{50}$$

From commutation relations (5), we obtain that the operators E_1, E_2, M_{12} satisfy the following commutation relations for the Lie algebra of $ISO(2)$

$$\begin{aligned} [E_1, E_2] &= 0, \\ [M_{12}, E_2] &= -iE_1, \\ [M_{12}, E_1] &= iE_2. \end{aligned} \tag{51}$$

These operators commute with the standard momentum matrix $\overset{\circ}{p}$ and the algebra itself is isomorphic to the Lie algebra of a small group of the momentum $\overset{\circ}{p}$.

Representations of the 1st type of the group $ISO(2)$ correspond to particles with continuous spin and therefore are not interesting from a physical point of view.[3)] And for representations of the 2nd type from (49) we have:

$$W_\mu = -\lambda P_\mu, \tag{52}$$

since for these representations E_1, E_2 - unit operators. Λ parameter is called helicity, and its absolute value is the spin of a massless particle. In this regard, it is necessary to make a few comments:

[3)] Theoretical studies devoted to particles with continuous spin have recently appeared [10].

- For the 2nd type representations of the group $ISO(2)$ group representations, the helicity λ takes integer values since on the Euclidean plane the angle of rotation changes in the interval $[0, 2\pi)$. But since in physics of elementary particles it is not the $ISO(3,1)$ Poincarè group itself that is used, but its universal covering group, the so-called Poincarè quantum-mechanical group $\overline{P_+^\uparrow}$ and its representations, for which rotation angle varies in the interval $[0, 4\pi)$. Accordingly, the helicity λ takes integer or half-integer values:

$$\lambda = 0, \pm 1/2, \pm 1, \pm 3/2, \ldots .$$

- For elementary particles whose states are transformed by irreducible representations of the group $\overline{P_+^\uparrow}$, there is one polarization state. But elementary particles like photon have two states of polarization. This means that elementary particles for which there are two independent states of polarization differing in the helicity sign λ are transformed according to irreducible representations of the Poincarè group with reflection.

APPENDIX

All calculations are based on formula (39). According to (25) for $h(\mathbf{p})$ we have

$$h(p, \vartheta, \varphi) = R(\varphi, \vartheta, 0) e^{\frac{1}{2}\sigma_3 \beta}. \tag{53}$$

Using the definition of Euler angles we get

$$R(\varphi, \vartheta, 0) = \begin{pmatrix} e^{-i\frac{\varphi}{2}} \cos\frac{\vartheta}{2} & -e^{-i\frac{\varphi}{2}} \sin\frac{\vartheta}{2} \\ e^{i\frac{\varphi}{2}} \sin\frac{\vartheta}{2} & e^{i\frac{\varphi}{2}} \cos\frac{\vartheta}{2} \end{pmatrix}. \tag{54}$$

From (53)-(54) we get the matrix expressions for $h(\mathbf{p})$ and $h^{-1}(\mathbf{p})$

$$\begin{aligned} h(\mathbf{p}) &= \begin{pmatrix} e^{-i\frac{\varphi}{2}} (\cosh\frac{\beta}{2} + \sinh\frac{\beta}{2}) \cos\frac{\vartheta}{2} & -e^{-i\frac{\varphi}{2}} (\cosh\frac{\beta}{2} - \sinh\frac{\beta}{2}) \sin\frac{\vartheta}{2} \\ e^{i\frac{\varphi}{2}} (\cosh\frac{\beta}{2} + \sinh\frac{\beta}{2}) \sin\frac{\vartheta}{2} & e^{i\frac{\varphi}{2}} (\cosh\frac{\beta}{2} - \sinh\frac{\beta}{2}) \cos\frac{\vartheta}{2} \end{pmatrix} = \\ &= \begin{pmatrix} e^{\frac{\beta - i\varphi}{2}} \cos\frac{\vartheta}{2} & -e^{-\frac{\beta + i\varphi}{2}} \sin\frac{\vartheta}{2} \\ e^{\frac{\beta + i\varphi}{2}} \sin\frac{\vartheta}{2} & e^{-\frac{\beta - i\varphi}{2}} \cos\frac{\vartheta}{2} \end{pmatrix}, \end{aligned} \tag{55}$$

$$h^{-1}(\mathbf{p}) = \begin{pmatrix} e^{-\frac{\beta-i\varphi}{2}}\cos\frac{\vartheta}{2} & e^{-\frac{\beta+i\varphi}{2}}\sin\frac{\vartheta}{2} \\ -e^{\frac{\beta+i\varphi}{2}}\sin\frac{\vartheta}{2} & e^{\frac{\beta-i\varphi}{2}}\cos\frac{\vartheta}{2} \end{pmatrix}. \tag{56}$$

From equations (28) and (55) it is clear that

$$p = h(\mathbf{p})h^{+}(\mathbf{p}).$$

The explicit expression of Wigner's rotation from (40) has the form

$$R(\eta_1,\omega,\eta_2) = \begin{pmatrix} e^{-i\frac{\eta_1+\eta_2}{2}}\cos\frac{\omega}{2} & -e^{-i\frac{\eta_1-\eta_2}{2}}\sin\frac{\omega}{2} \\ e^{i\frac{\eta_1-\eta_2}{2}}\sin\frac{\omega}{2} & e^{i\frac{\eta_1+\eta_2}{2}}\cos\frac{\omega}{2} \end{pmatrix}. \tag{57}$$

Inverse formulas for β,ϑ,φ can be found using (25)-(27).
Now we calculate the Wigner rotation in special cases of A:

1. $A = e^{-i\frac{1}{2}\sigma_2\psi} = \begin{pmatrix} \cos\frac{\psi}{2} & -\sin\frac{\psi}{2} \\ \sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{pmatrix}.$

   Then from (15) and (28) we have

$$\begin{aligned} \beta' &= \beta, \\ \sin\vartheta'\sin\varphi' &= \sin\vartheta\sin\varphi, \\ \sin\vartheta'\cos\varphi' &= \cos\vartheta\sin\psi + \sin\vartheta\cos\varphi\cos\psi, \\ \cos\vartheta' &= \cos\vartheta\cos\psi - \sin\vartheta\cos\varphi\sin\psi. \end{aligned} \tag{58}$$

   Comparing the expression for the matrix A and the formulas (55)-(58) and (39) we get

$$\begin{aligned} e^{-i\frac{\eta_1+\eta_2}{2}}\cos\frac{\omega}{2} = e^{i\frac{\varphi'}{2}}\left(e^{-i\frac{\varphi}{2}}\cos\frac{\psi}{2}\cos\frac{\vartheta}{2} - e^{i\frac{\varphi}{2}}\sin\frac{\psi}{2}\sin\frac{\vartheta}{2}\right)\cos\frac{\vartheta'}{2} + \\ + e^{-i\frac{\varphi'}{2}}\left(e^{-i\frac{\varphi}{2}}\sin\frac{\psi}{2}\cos\frac{\vartheta}{2} + e^{i\frac{\varphi}{2}}\cos\frac{\psi}{2}\sin\frac{\vartheta}{2}\right)\sin\frac{\vartheta'}{2}, \end{aligned} \tag{59}$$

$$e^{i\frac{\eta_1+\eta_2}{2}}\cos\frac{\omega}{2} = e^{-i\frac{\varphi'}{2}}\left(e^{i\frac{\varphi}{2}}\cos\frac{\psi}{2}\cos\frac{\vartheta}{2} - e^{-i\frac{\varphi}{2}}\sin\frac{\psi}{2}\sin\frac{\vartheta}{2}\right)\cos\frac{\vartheta'}{2} + \\ + e^{i\frac{\varphi'}{2}}\left(e^{i\frac{\varphi}{2}}\sin\frac{\psi}{2}\cos\frac{\vartheta}{2} + e^{-i\frac{\varphi}{2}}\cos\frac{\psi}{2}\sin\frac{\vartheta}{2}\right)\sin\frac{\vartheta'}{2}, \quad (60)$$

$$e^{i\frac{\eta_1-\eta_2}{2}}\sin\frac{\omega}{2} = e^{\beta-i\frac{\varphi'}{2}}\left(e^{-i\frac{\varphi}{2}}\sin\frac{\psi}{2}\cos\frac{\vartheta}{2} + e^{i\frac{\varphi}{2}}\cos\frac{\psi}{2}\sin\frac{\vartheta}{2}\right)\cos\frac{\vartheta'}{2} - \\ - e^{\beta+i\frac{\varphi'}{2}}\left(e^{-i\frac{\varphi}{2}}\cos\frac{\psi}{2}\cos\frac{\vartheta}{2} - e^{i\frac{\varphi}{2}}\sin\frac{\psi}{2}\sin\frac{\vartheta}{2}\right)\sin\frac{\vartheta'}{2}, \quad (61)$$

$$e^{-i\frac{\eta_1-\eta_2}{2}}\sin\frac{\omega}{2} = e^{-\beta+i\frac{\varphi'}{2}}\left(e^{i\frac{\varphi}{2}}\sin\frac{\psi}{2}\cos\frac{\vartheta}{2} + e^{-i\frac{\varphi}{2}}\cos\frac{\psi}{2}\sin\frac{\vartheta}{2}\right)\cos\frac{\vartheta'}{2} - \\ - e^{-\beta-i\frac{\varphi'}{2}}\left(e^{i\frac{\varphi}{2}}\cos\frac{\psi}{2}\cos\frac{\vartheta}{2} - e^{-i\frac{\varphi}{2}}\sin\frac{\psi}{2}\sin\frac{\vartheta}{2}\right)\sin\frac{\vartheta'}{2}. \quad (62)$$

From equations (61)-(62) using complex conjugation, you can get

$$e^{-\beta+i\frac{\eta_1-\eta_2}{2}}\sin\frac{\omega}{2} = e^{\beta+i\frac{\eta_1-\eta_2}{2}}\sin\frac{\omega}{2}.$$

The last equality must be fulfilled for all real β. And this is only possible with $\omega = 0$.[4] It follows that the rotation (57) depends only on the angular parameter $\eta \equiv \eta_1 + \eta_2$.

From the equations (59)-(60) you can get the following:

$$e^{i\frac{\eta}{2}} = \left(e^{i\frac{\varphi-\varphi'}{2}}\cos\frac{\psi}{2}\cos\frac{\vartheta}{2} - e^{-i\frac{\varphi'+\varphi}{2}}\sin\frac{\psi}{2}\sin\frac{\vartheta}{2}\right)\cos\frac{\vartheta'}{2} + \\ + \left(e^{i\frac{\varphi'+\varphi}{2}}\sin\frac{\psi}{2}\cos\frac{\vartheta}{2} + e^{i\frac{\varphi'-\varphi}{2}}\cos\frac{\psi}{2}\sin\frac{\vartheta}{2}\right)\sin\frac{\vartheta'}{2}. \quad (63)$$

From the last system of equations, you can get the following:

$$\cos\frac{\eta}{2} = \left(\cos\frac{\varphi'-\varphi}{2}\cos\frac{\psi}{2}\cos\frac{\vartheta}{2} - \cos\frac{\varphi'+\varphi}{2}\sin\frac{\psi}{2}\sin\frac{\vartheta}{2}\right)\cos\frac{\vartheta'}{2} + \\ + \left(\cos\frac{\varphi'+\varphi}{2}\sin\frac{\psi}{2}\cos\frac{\vartheta}{2} + \cos\frac{\varphi'-\varphi}{2}\cos\frac{\psi}{2}\sin\frac{\vartheta}{2}\right)\sin\frac{\vartheta'}{2}. \quad (64)$$

[4] Recall that the Euler angle ω varies within $0 \le \omega \le \pi$.

$$\sin\frac{\eta}{2} = \left(-\sin\frac{\varphi'-\varphi}{2}\cos\frac{\psi}{2}\cos\frac{\vartheta}{2} + \sin\frac{\varphi'+\varphi}{2}\sin\frac{\psi}{2}\sin\frac{\vartheta}{2}\right)\cos\frac{\vartheta'}{2} + \\ + \left(\sin\frac{\varphi'+\varphi}{2}\sin\frac{\psi}{2}\cos\frac{\vartheta}{2} + \sin\frac{\varphi'-\varphi}{2}\cos\frac{\psi}{2}\sin\frac{\vartheta}{2}\right)\sin\frac{\vartheta'}{2}. \quad (65)$$

And the element of a small group (57) takes the following form

$$R = \begin{pmatrix} e^{-i\frac{\eta}{2}} & -e^{-i\frac{\eta}{2}} \\ e^{i\frac{\eta}{2}} & e^{i\frac{\eta}{2}} \end{pmatrix}. \quad (66)$$

After the necessary transformations from (64)-(65), we get[5)]

$$2\cos\eta = \cos\psi\left(\sin\vartheta\sin\vartheta' + \sin\varphi\sin\varphi'\right) + \\ + \cos\vartheta'\left(\cos\vartheta\sin\varphi\sin\varphi' - \cos\varphi'\sin\vartheta\sin\psi\right) + \\ + \cos\varphi\left(\cos\varphi' + \cos\vartheta\cos\vartheta'\cos\varphi'\cos\psi + \cos\vartheta\sin\vartheta'\sin\psi\right)$$

Hence, with the help of (58), we finally

$$\cos\eta = \frac{\cos\psi - \cos\vartheta\cos\vartheta'}{\sin\vartheta\sin\vartheta'}; \qquad \omega = 0. \quad (67)$$

2. $A = e^{-i\frac{1}{2}\sigma_3\psi} = \begin{pmatrix} e^{-i\frac{1}{2}\sigma_3\psi} & 0 \\ 0 & e^{i\frac{1}{2}\sigma_3\psi} \end{pmatrix}.$

For this case from (15) and (28) we have:

$$\beta' = \beta, \quad \vartheta' = \vartheta, \quad \varphi' = \varphi + \psi.$$

Comparing the expression for the matrix A and formulas (55)-(57) and (39) we get

$$e^{i\frac{\eta_1-\eta_2}{2}}\sin\frac{\omega}{2} = 0, \quad e^{i\frac{\eta_1+\eta_2}{2}}\cos\frac{\omega}{2} = 1. \quad (68)$$

Hence, as in the previous case, we obtain

$$\eta_1 = \eta_2 = \omega = 0. \quad (69)$$

So in this case, Wigner's rotation does not occur.

5) It is convenient to use the identity here: $\cos\eta = \cos^2\frac{\eta}{2} - \sin^2\frac{\eta}{2}$.

3. $A = e^{\frac{1}{2}\sigma_3\alpha} = \begin{pmatrix} \cosh\frac{\alpha}{2} + \sinh\frac{\alpha}{2} & 0 \\ 0 & \cosh\frac{\alpha}{2} - \sinh\frac{\alpha}{2} \end{pmatrix} = \begin{pmatrix} e^{\alpha/2} & 0 \\ 0 & e^{-\alpha/2} \end{pmatrix}.$

In this case from (15) and (28) we have

$$\begin{aligned} \cosh\beta' &= \cosh\beta\cosh\alpha + \sinh\beta\sinh\alpha\cos\vartheta; \\ \sinh\beta'\sin\vartheta' &= \sinh\beta\sin\vartheta; \\ \sinh\beta'\cos\vartheta' &= \cosh\beta\sinh\alpha + \sinh\beta\cosh\alpha\cos\vartheta; \\ \varphi' &= \varphi. \end{aligned} \tag{70}$$

Comparing the expression for the matrix A and the formulas (55)-(57) and (39) we get

$$\begin{aligned} e^{-i\frac{\eta_1+\eta_2}{2}}\cos\frac{\omega}{2} &= e^{\frac{\beta-\alpha-\beta'}{2}}\sin\frac{\vartheta}{2}\sin\frac{\vartheta'}{2} + e^{\frac{-\beta'+\alpha+\beta}{2}}\cos\frac{\vartheta}{2}\cos\frac{\vartheta'}{2}, \\ e^{-i\frac{\eta_1-\eta_2}{2}}\sin\frac{\omega}{2} &= e^{\frac{-\beta'+\alpha-\beta}{2}}\sin\frac{\vartheta}{2}\cos\frac{\vartheta'}{2} - e^{\frac{-\beta'-\alpha-\beta}{2}}\cos\frac{\vartheta}{2}\sin\frac{\vartheta'}{2}, \\ e^{i\frac{\eta_1-\eta_2}{2}}\sin\frac{\omega}{2} &= e^{\frac{\beta'-\alpha+\beta}{2}}\sin\frac{\vartheta}{2}\cos\frac{\vartheta'}{2} - e^{\frac{\beta'+\alpha+\beta}{2}}\cos\frac{\vartheta}{2}\sin\frac{\vartheta'}{2}, \\ e^{i\frac{\eta_1+\eta_2}{2}}\cos\frac{\omega}{2} &= e^{\frac{\beta'+\alpha-\beta}{2}}\sin\frac{\vartheta}{2}\sin\frac{\vartheta'}{2} + e^{\frac{\beta'-\alpha-\beta}{2}}\cos\frac{\vartheta}{2}\cos\frac{\vartheta'}{2}. \end{aligned} \tag{71}$$

From (70)-(71) after the necessary calculations, one can find expressions for Euler angles

$$\begin{aligned} \eta_1 &= \eta_2 = 0, \\ \cos\omega &= \cos\vartheta\cos\vartheta' + \cosh\alpha\sin\vartheta\sin\vartheta' = \\ &= \frac{\cosh\beta\cosh\beta' - \cosh\alpha}{\sinh\beta\sinh\beta'}. \end{aligned} \tag{72}$$

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