GEOMETRIC INTERPRETATION OF WIGNER SPIN ROTATION

B. A. Rajabov*

Shamakhy Astrophysical Observatory named after Nasireddin Tusi of the Azerbaijan National Academy of Sciences

This paper is devoted to the application of the Wigner method of constructing representations of the quantum-mechanical Poincaré group $ISO(3.1)$ using the small group method and its application in the theory of elementary particles. Explicit formulas are given for the angles of rotation of the spin during Lorentz turns, and geometric interpretations of the results of Wigner's theory in terms of spherical and hyperbolic geometries are given. The results admit a direct generalization to the cosmological groups $SO(4,1)$ and $SO(3,2)$.
Keywords: Poincarè group $ISO(3,1)$– spherical geometry– hyperbolic geometry–unitary ray representations–small group–Wigner's operator.

1. INTRODUCTION

In this paper, we consider the Wigner spin rotation under Lorentz transformations. Full details of the article are in the review [1].

Wigner rotation is determined from the following formula:

$$\tilde{A}(\mathring{p}, A) = h^{-1}(\mathbf{p}')Ah(\mathbf{p}). \tag{1}$$

Here $A \in SL(2,C)$ and $h(\mathbf{p} \in SL(2,C)$ are the Lorentz transform and the Wigner operator ("boost"), respectively; $SL(2,C)$ is the group of complex second-order unimodular matrices, which is the universally covering group of the Lorentz group $SO(3,1)$. The matrix $\tilde{A}(\mathring{p}, A)$ defines the rotation of the rest coordinate system:

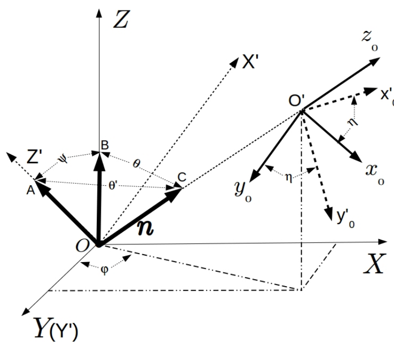
$$\tilde{A}(\mathring{p}, A) \equiv R(\eta_1, \omega, \eta_2) = e^{-i\frac{1}{2}\sigma_3\eta_1} e^{-i\frac{1}{2}\sigma_2\omega} e^{-i\frac{1}{2}\sigma_3\eta_2}. \tag{2}$$

Here η_1, ω, η_2 are Euler angles (in this case Wigner angles!) of rotation R.

We will consider 3 different cases of the matrix A since all other cases are a combination of them.

1. Let A be the rotation around the y axis: $A = e^{-i\frac{1}{2}\sigma_2\psi}$.

* E-mail: balaali.rajabov@mail.ru



$$\mathbf{n} = \frac{\mathbf{p}}{|\mathbf{p}|}$$
$$\mathbf{y}_0 \perp \mathbf{z}\,\mathbf{z}_0$$
$$\mathbf{y}_0^{'} \perp \mathbf{z}^{'}\mathbf{z}_0$$

Fig. 1.

The rotation of the rest coordinate system is given by the formulas:

$$\cos\eta = \frac{\cos\psi - \cos\vartheta\cos\vartheta'}{\sin\vartheta\sin\vartheta'}; \quad \omega = 0. \tag{3}$$

The expression(3) has a simple geometric interpretation (see Fig.1).

$$\cos C \equiv \cos\psi = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos\eta.$$

In this case, we have the cosine theorem of spherical geometry.

2. Let A be a rotation around the axis z: $A = e^{-i\frac{1}{2}\sigma_3\psi}$.

In this case, the Wigner angles are zero: $\eta_1 = \eta_2 = \omega = 0$.

3. Let A represent the Lorentz transformation in the positive direction of the z-axis:

$$A = e^{\frac{1}{2}\sigma_3\alpha}.$$

In this case, the Wigner angles are given by the following formulas:

$$\cos\omega = \frac{\cosh\beta\cosh\beta' - \cosh\alpha}{\sinh\beta\sinh\beta'}; \qquad \eta_1 = \eta_2 = 0. \tag{4}$$

$$\cosh\beta = \frac{p_0}{m}, \qquad \sinh\beta = \frac{|\mathbf{p}|}{m}; \qquad \cosh\beta' = \frac{p_0^{'}}{m}, \qquad \sinh\beta' = \frac{|\mathbf{p}^{'}|}{m} \tag{5}$$

It can be shown that the expression (4) has an geometric interpretation. We introduce 4-speeds:

$$u_\mu = \frac{p_\mu}{m} = \left(\frac{p_0}{m}, -\frac{\mathbf{p}}{m}\right),$$
$$u_\mu u^\mu = \frac{p_\mu p^\mu}{m^2} = 1$$

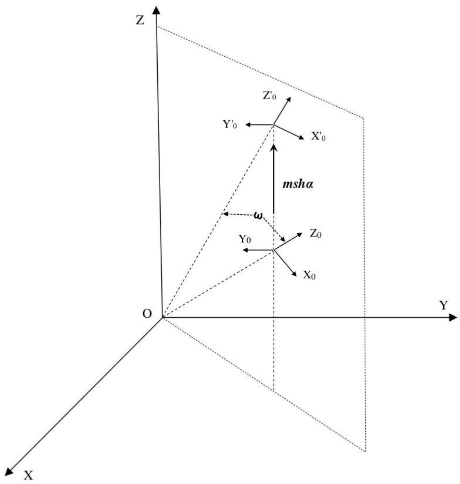


Fig. 2.

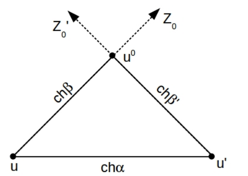


Fig. 3.

The last equality means that in the 4-velocity space the ends of the 4-vectors lie on the surface of the hyperboloid (the point $(1, 0, 0, 0)$ is the vertex of the hyperboloid). Obviously, the end of the vector $\overset{\circ}{u} = (1, 0, 0, 0)$ will be at the vertex of the hyperboloid. Consider a triangle formed by vertices at the points $\overset{\circ}{u}, u, u'$ (Fig. 3).

$$u^\mu \overset{\circ}{u}_\mu = \overset{\circ}{u} = \cosh\beta; \quad u'^\mu \overset{\circ}{u}_\mu = \cosh\beta'; \quad u^\mu u'_\mu \equiv \cosh\alpha. \qquad (6)$$

If we apply the cosine theorem of hyperbolic geometry for this triangle, we get:

$$\cosh\alpha \equiv u^\mu u'_\mu = \cosh\beta \cosh\beta' - \sinh\beta \sinh\beta' (\mathbf{n}\mathbf{n}').$$

On the other hand, the angle between $\mathbf{n}'$ and $\mathbf{n}$ is the angle of rotation around the axis y_0:

$$(\mathbf{n}\,\mathbf{n}') = \cos\omega$$

Finally, we get:

$$\cosh\alpha = \cosh\beta \cosh\beta' - \sinh\beta \sinh\beta' \cos\omega \qquad (7)$$

It should be noted that the above applies to the case of massive particles:

$$m^2 > 0.$$

REFERENCES

1. Rajabov B.A., Spin rotation under Lorentz transformations and its geometric interpretations, **14**(2), 2019. (to be printed)